7. Static Equilibria and Systems of Equations

We'll now use some of the ideas we've encountered to practice mathematical economics (i.e., the use of mathematical methods/techniques to construct rigorous/logical economic theories that can be tested), beginning where most first econ courses begin: analyzing equilibria in markets.

Since "equilibrium" is a tendency not to change, the study of equilibria is sometimes called "statics." **The standard static equilibrium problem is to find values of the endogenous variables, given the structure of the model, that satisfy the equilibrium condition.** We'll start with a "**partial equilibrium**" problem—a price determination in an isolated market, where everything else is considered exogenous and fixed, unchanging as our market adjusts—and work toward a "**general equilibrium**" problem, in which several markets interact simultaneously.

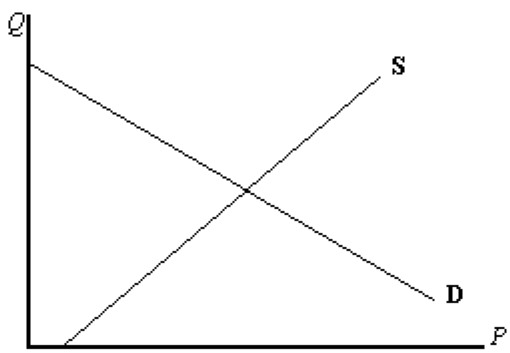
a. Partial Equilibrium

Given

an equilibrium condition:	$Q_d = Q_s$		2.12
a demand curve:	$Q_d = a - b \cdot P$,	$a, b > 0$	2.13
and a supply curve:	$Q_s = -c + d \cdot P$	$c, d > 0$	2.14

we can solve for solution values of $\overline{P}$ and $\overline{Q}$, stated in terms of the parameters and exogenous variables of the model. (The negative intercept in 2.14 assures that quantity supplied will be zero unless price is sufficiently high.)

In pictures, we'd have:



(Notice that I've placed the dependent variable, quantity, on the vertical axis.) Using algebra, we can insert 2.13 and 2.14 into 2.12 to yield

$$a - b\overline{P} = -c + d\overline{P} \qquad 2.15$$

or, solving for quilibrium price,

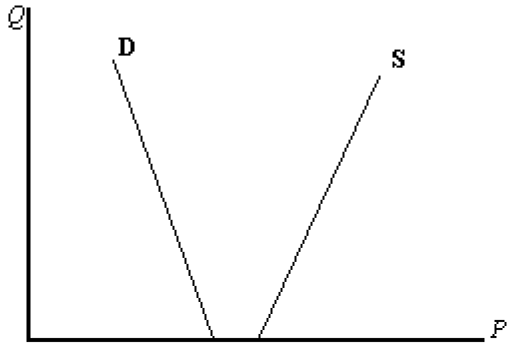
$$\overline{P} = \frac{a+c}{b+d} \qquad 2.16$$

We can insert 2.16 into either 2.13 or 2.14 to find equilibrium quantity,

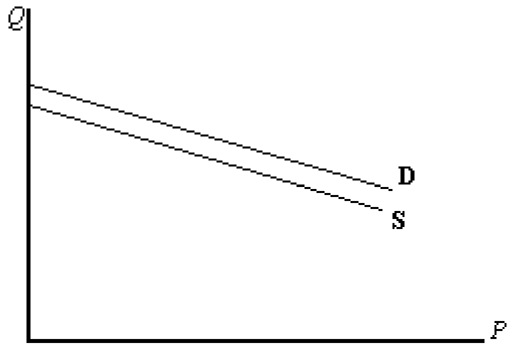
$$\overline{Q} = \frac{ad - bc}{b+d} \quad . \qquad 2.17$$

Some notes:

- Since $(b+d) > 0$ by assumption (both *b* and *d* exceed zero), in order for 2.17 to make sense, *ad* must normally exceed *bc*. Otherwise we have a negative equilibrium quantity, a "disposal solution:"



- If $b+d=0$, violating our assumption that both *b* and *d* exceed zero, there is no solution:



- If we simultaneously violate the assumption that *a* and *c* exceed zero, the two curves could lie on top of each other (that is, be *coincident*, or *linearly dependent*.

b. General Equilibrium

Let's do the simplest case: two goods in two markets, with linear demand and supply functions:

Good #1:		Good #2:	
$Q_{d1} - Q_{s1} = 0$	2.18	$Q_{d2} - Q_{s2} = 0$	2.21
$Q_{d1} = a_0 + a_1 P_1 + a_2 P_2$	2.19	$Q_{d2} = \alpha_0 + \alpha_1 P_1 + \alpha_2 P_2$	2.22
$Q_{s1} = b_0 + b_1 P_1 + b_2 P_2$	2.20	$Q_{s2} = \beta_0 + \beta_1 P_1 + \beta_2 P_2$	2.23

Substitution 2.19 and 2.20 into 2.18, and 2.22 and 2.23 into 2.21, yields two equations in two unknown equilibrium prices:

$$(a_0 - b_0) + (a_1 - b_1)\overline{P}_1 + (a_2 - b_2)\overline{P}_2 = 0 \qquad 2.24$$

$$(\alpha_0 - \beta_0) + (\alpha_1 - \beta_1)\overline{P}_1 + (\alpha_2 - \beta_2)\overline{P}_2 = 0 \qquad 2.25$$

To simplify the notation, let $c_i = a_i - b_i$, $\gamma_i = \alpha_i - \beta_i$, where $i = 0,1,2.$ Equations 2.24 and 2.25 become

$$c_1 \overline{P}_1 + c_2 \overline{P}_2 = -c_0 \qquad 2.26$$

$$\gamma_1 \overline{P}_1 + \gamma_2 \overline{P}_2 = -\gamma_0 \qquad 2.27$$

Solving 2.26 for $\overline{P}_1$ and substituting into 2.27 yields the equilibrium prices in terms of the model parameters alone:

$$\overline{P}_1 = \frac{c_2\gamma_0 - c_0\gamma_2}{c_1\gamma_2 - c_2\gamma_1} \qquad 2.28$$

$$\overline{P}_2 = \frac{c_0\gamma_1 - c_1\gamma_0}{c_1\gamma_2 - c_2\gamma_1} \qquad 2.29$$

We can use these equilibrium prices to find the equilibrium quantities from 2.18 through 2.23.

There are elegant proofs outlining the requirements for existance and uniqueness of the solutions to such systems; the matrix algebra proofs for more than two dimensions place conditions on the matrix determinants in the linear case, and on the jacobian determinants for the nonlinear case. In general, two things are demonstrated by these proofs: To yield a unique solution, the equations must be consistent (for example, nothing like $x+y=8$, $x+y=9$) and must be functionally independent (for example, nothing like $x+y=2$, $2x+2y=4$). Each equation must give information that is not contradicted by the others, and not already contained in the others.

8. Review of Calculus with Applications to Reduced-Form Comparative Statics

a. Rules of Differentiation

One function, one variable:

1. Constant function: If $y = f(x) = k$, then

$$\frac{dy}{dx} = f'(x) = 0 \qquad 2.30$$

2. Power function: If $y = ax^n$, then

$$f'(x) = nx^{n-1} \qquad 2.31$$

Two functions, one variable:

3. Sum-Difference: $$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) = f'(x) \pm g'(x) \qquad 2.32$$

4. Product: $$\frac{d}{dx}[f(x)\cdot g(x)] = f(x)g'(x) + g(x)f'(x) \qquad 2.33$$

5. Quotient: $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} \qquad 2.34$$

Two functions, two variables:

6. Chain: If $z = f(y)$ and $y = g(x)$, then

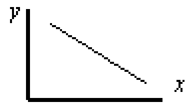
$$\frac{dz}{dx} = \frac{dz}{dy}\cdot\frac{dy}{dx} = f'(y)g'(x) \qquad 2.35$$

7. Inverse Function Rule:

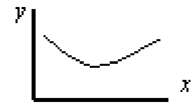
Existence: If *y* is a function of *x*, $y = f(x)$, when may we also consider *x* to be a function of *y*, $x = f^{-1}(y)$, read "*x* is an inverse function of *y*?"

Answer: If and only if $f(x)$ is monotonic, which we can check by finding the first derivative and checking to see that it does not change sign over the function's domain. 2.36

Examples:



Yes



No

Derivative of $x = f^{-1}(y)$: If $y = f(x)$, and $x = f^{-1}(y)$, then

$$\frac{dx}{dy} = \frac{1}{dy/dx} \qquad 2.37$$

Example: If $y = x^9 + x^5 + 17$, it's difficult to find $\frac{dx}{dy}$ by solving for *x*, but easy using the inverse function rule:

$$\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{9x^8 + 5x^4}.$$

8. Partial differentiation: Given $y = f(x_1, x_2, \ldots, x_n)$, where all x_i are independent, then $\frac{\partial y}{\partial x_i} = f_i(x)$, read "the partial derivative of *y* with respect to x_i." All other x_i are treated as constants when differentiating. 2.38

Example: Given $R = P \cdot Q = P(a - bP + cA)$, find $\frac{\partial R}{\partial A}$.

Using partial derivatives and following the chain rule, we have $\frac{\partial R}{\partial A} = Pc$.

b. Applications to Reduced-Form Comparative Statics

Here we treat all parameters as independent. We "boil the system of equations down" to a "reduced form solution"—solutions for the endogenous variables that involve only explicit functions of the parameters and exogenous variables. Then we take partial derivatives of these reduced form equations, to find the "comparative static" effects of changes in parameters and exogenous variables.

A simple microeconomic example: Say we recall our earlier system, with

an equilibrium condition: $Q_d = Q_s$ 2.12

a demand curve: $Q_d = a - b \cdot P$, $a, b > 0$ 2.13

and a supply curve: $Q_s = -c + d \cdot P$ $c, d > 0$ 2.14 .

This yielded the following reduced form solution:

$$\overline{P} = \frac{a + c}{b + d} \qquad 2.16$$

$$\overline{Q} = \frac{ad - bc}{b + d} \qquad 2.17$$

Now we can take partial derivatives to see how the equilibrium solution will change if the model's parameters change:

$$\frac{\partial \overline{P}}{\partial a} = \frac{1}{b+d} > 0 \qquad 2.39$$

In words: If the intercept of the demand curve increases, so will the equilibrium price.
We also have

$$\frac{\partial \overline{P}}{\partial c} = \frac{1}{b+d} > 0\,, \qquad 2.30$$

and

$$\frac{\partial \overline{P}}{\partial b} = \frac{\partial \overline{P}}{\partial d} = \frac{-(a+c)}{(b+d)^2} < 0 \qquad 2.31$$