

## Chapter 2: Whirlwind Tour of Mathematical Economics

### Economic Modeling, Static Equilibria and Systems of Equations

#### 1. This Chapter is Special

Strictly speaking, “mathematical economics” is a sub-specialty within the profession, separate from the econometrics sub-specialty. Mathematical economics centers around expressing and analyzing economic theories through mathematical symbols; econometrics involves measuring these symbols’ representatives in the real world-- using or evaluating the fruits of mathematical economics.

Econometrics is therefore dependent upon mathematical economics in some ways, so it’s a good idea to understand some of the elements of mathematical economics before proceeding into econometrics. This chapter will review those elements. For a more detailed treatment, I’d recommend *Fundamental Methods of Mathematical Economics* by Alpha C. Chiang or, for those with strong preparation in matrix algebra, *Mathematics for Economists* by Carl P. Simon and Lawrence Blume.

This text is founded on the proposition that some things are best done as pre-class text readings, and others are best done in person. In my experience, it’s wise to discuss the topics of this chapter personally. So I have provided an outline of the topics for you to look through before class.

#### 2. General advantages/disadvantages of the “mathematical economics” approach

:

- a. It forces one to have some explicit theory.  
“Knowledge is always gained by the orderly loss of information.”
  - ai. Often only the mathematical assumptions (linearity, homogeneity) are made explicit, leaving the economic assumptions (rationality, perfect information) buried.
  - aii. The method can be overplayed; elegant, empty constructs with no policy usefulness.
  - aiii. Simplifying assumptions are often lost/forgotten later, amidst the details.
- b. Math frequently yields results that are more general: a whole class of problems may be solved at once  
lucid: some concepts are more understandable using math (marginal product, risk, imperfect information)  
tractable: some problems can’t be solved without math (effects of various taxes or subsidies, results of foreign trade)
  - bi. Mathematical models may ignore qualitative variables (attitudes, beliefs, class), not because they are unimportant, but because they’re not amenable to mathematical analysis.
  - bii. Even where useful for analysis, mathematics can’t generate values or decide what is important, which should be part of the work of economists

#### 3. Economic Modeling

A **model** is a deliberately simplified picture/framework/approximation/abstraction of the part of the world we’re out to analyze.

A **mathematical model** is therefore usually a series of **equations** that assume/impose some structure on the world, a relationship among **variables** being studied.

In such models there are generally **three types of equations and three kinds of variables**. The equations can take on a variety of **functional forms**.

## 4. Types of Equations

**a. Definitional equations:** They give an identity; the two expressions have exactly the same meaning:

e.g.,  $\Pi = R - C$  2.1

or  $MC = \frac{dC}{dQ}$  2.2

or  $S_{\text{market}} = \sum MC_{\text{firms}}$  . 2.3

**b. Behavioral equations:** They report the way in which one variable responds to changes in another; they include an implicit description of the institutional setting of the model—technological capabilities, legal codes, tax structures, moral attitudes, and so forth.

For example, to begin breathing life into the first identity above, we could have

$C = 75 + 10*Q$  2.4

or  $C = 111 + Q$ . 2.5

Each of these represents a different institutional structure—in this case, technological and tax differences that yield higher fixed costs and lower marginal costs in the second equation.

So it's mainly through the specification of the behavioral equations that one gives life to the assumptions of a model. And now the punch line: Econometrics will allow us to measure and test these assumptions, by measuring/estimating parameters in behavioral equations!

Here are more examples of behavioral equations:

$R = \$10 * Q$  .(This assumes pure competition. Do you know why?) 2.6

or  $Q_d = 500 - 20P$  . 2.7

(2.7 is a linear market demand curve; it assumes nothing affects demand but the product's price.)

## c. Equilibrium Conditions

These appear only if the model involves some notion of equilibrium—an ultimate “rest point” of the system.

For example, put together the first equations of the last few sections,

$\Pi = R - C$  (a definition)

$C = 75 + 10*Q$  (a behavioral equation)

$R = \$10 * Q$  (yet another behavioral equation, consistent with pure competition),

and now add the long-run equilibrium condition of pure competition,

$\Pi = 0$  (zero economic profit), 2.8

and Voila, mes amis, a complete model of a **purely-competitive firm** in the long run!

If you gather together some of the other equations from above, you can construct a **short-run** model of a **purely competitive market**:

$C = 75 + 10*Q$

$MC = dc/dq$  (=10)

$S_{\text{market}} = \sum MC_{\text{firms}}$  (=10)(Those three equations together generate the market supply curve.)

$Q_d = 500 - 20P$  (There's the market demand curve.)

$Q_d = Q_s$  (There's the equilibrium condition—the market must clear.)

(Now, why is this only a short-run model? And why is it a model of pure competition?)

## 5. Types of Variables:

**a. Endogenous:** Variables whose value/solution we are seeking from the model.

**b. Exogenous:** Variables determined by forces outside what we are modeling at the moment—these are accepted as “given.” Often a subscript “o” is used to identify them.

Variables that are exogenous in one model may be endogenous in another—it all depends on how the analyst is abstracting from reality in one particular situation. For example, in the purely-competitive market example above, Price is endogenous. But if we were modeling a particular firm or consumer within this market, Price would be exogenous—the firm/consumer can't affect price; price is determined outside of the thing being modeled.

**c. Parameters:** A parameter is, in effect, a constant that is being treated as a variable! For example, say we rewrite the cost function

$$C = 75 + 10 \cdot Q$$

as  $C = F + m \cdot Q$ .

2.9

It's still a cost function, but now it includes both variables (C and Q) and parameters (F and m).

So coefficients are symbols that raise models to a higher level of generality: The second cost function here doesn't try to answer how costs respond to changes in quantity, in any particular circumstance in the real world.

## 6. Functional forms:

Most of the equations we will work with will be functions, relations between two variables in which each value of the "x" ("independent") variable is associated with a unique value of the "y" (dependent) variable. (Pause: Have all of the equations so far been functions?)

These functional relationships fall into a number of general types, each of which can be generalized into more than two dimensions:

**a. Constant functions:** e.g.,  $y = f(x) = 10$  (e.g., the marginal cost function earlier)

### b. Polynomial (literally "multi-term") functions

Here each term contains a coefficient on a non-negative integer power of the variable.

e.g.,  $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n$  2.10

Constant function \_\_\_\_\_ (straight line, zero slope)

Linear f'c'n \_\_\_\_\_ (straight line)

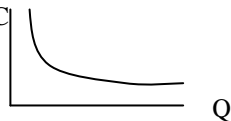
Quadratic f'c'n \_\_\_\_\_ (parabola)

Cubic f'c'n \_\_\_\_\_ (can have two "humps:" )

The value of "n" is the "degree" of the polynomial; rarely is  $n > 4$  (a "quartic" function) in economics.

**c. Rational functions:** These are functions that declined admission to Hope College. Here y is the ratio of two polynomials in x.

e.g.,  $y = a/x$ , the rectangular hyperbola, is a rational; its the traditional form of the AFC cost function: AFC

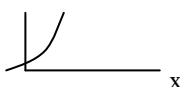


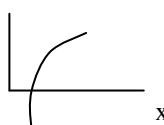
Let  $C = 75 + 10 \cdot Q$ .  
Then  $FC = 75$   
and  $AFC = 75/Q$

Another e.g.: The "constant elasticity of demand" demand function:

$$Q = [a_0 + a_1P_s + a_2Y] / P \quad (\text{implies price elasticity of demand} = 1) \quad 2.11$$

**d. Nonalgebraic (Transcendental) functions:** These functions can't be expressed as polynomials or roots of polynomials. The most common classes of transcendentals for us will be:

Exponential functions:  $y = b^x$  

Logarithmic functions:  $y = \log_b x$  

(Pause: Write a linear function in three dimensions. Now picture this function in space. Now make it a third-degree polynomial, and picture how it changes in space. Now add a fourth dimension.)

## 7. Static Equilibria and Systems of Equations

We'll now use some of the ideas we've encountered to practice mathematical economics (i.e., the use of mathematical methods/techniques to construct rigorous/logical economic theories that can be tested), beginning where most first econ courses begin: analyzing equilibria in markets.

Since "equilibrium" is a tendency not to change, the study of equilibria is sometimes called "statics." **The standard static equilibrium problem is to find values of the endogenous variables, given the structure of the model, that satisfy the equilibrium condition.** We'll start with a "**partial equilibrium**" problem—a price determination in an isolated market, where everything else is considered exogenous and fixed, unchanging as our market adjusts—and work toward a "**general equilibrium**" problem, in which several markets interact simultaneously.

### a. Partial Equilibrium

Given

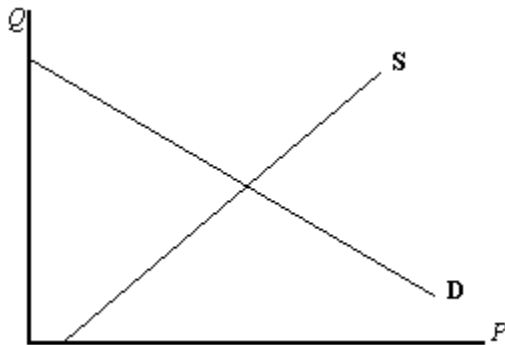
$$\text{an equilibrium condition:} \quad Q_d = Q_s \quad 2.12$$

$$\text{a demand curve:} \quad Q_d = a - b \cdot P, \quad a, b > 0 \quad 2.13$$

$$\text{and a supply curve:} \quad Q_s = -c + d \cdot P \quad c, d > 0 \quad 2.14$$

we can solve for solution values of  $\bar{P}$  and  $\bar{Q}$ , stated in terms of the parameters and exogenous variables of the model. (The negative intercept in 2.14 assures that quantity supplied will be zero unless price is sufficiently high.)

In pictures, we'd have:



(Notice that I've placed the dependent variable, quantity, on the vertical axis.) Using algebra, we can insert 2.13 and 2.14 into 2.12 to yield

$$a - b\bar{P} = -c + d\bar{P} \quad 2.15$$

or, solving for equilibrium price,

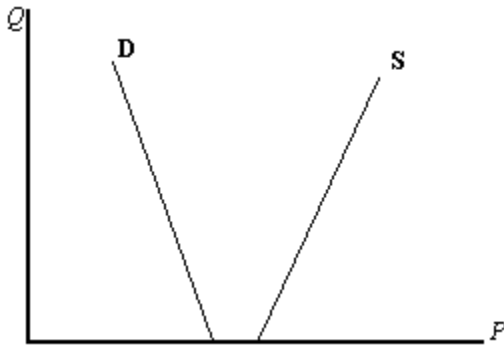
$$\bar{P} = \frac{a + c}{b + d} \quad 2.16$$

We can insert 2.16 into either 2.13 or 2.14 to find equilibrium quantity,

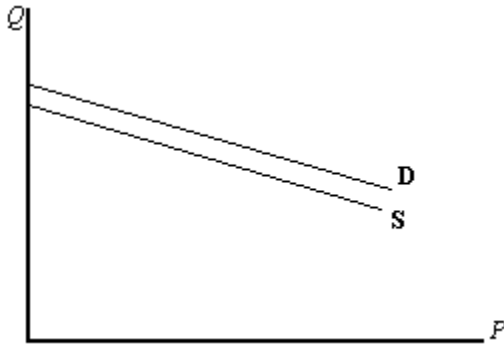
$$\bar{Q} = \frac{ad - bc}{b + d} \quad 2.17$$

Some notes:

- Since  $(b + d) > 0$  by assumption (both  $b$  and  $d$  exceed zero), in order for 2.17 to make sense,  $ad$  must normally exceed  $bc$ . Otherwise we have a negative equilibrium quantity, a "disposal solution."



- If  $b + d = 0$ , violating our assumption that both  $b$  and  $d$  exceed zero, there is no solution:



- If we simultaneously violate the assumption that  $a$  and  $c$  exceed zero, the two curves could lie on top of each other (that is, be *coincident*, or *linearly dependent*).

### b. General Equilibrium

Let's do the simplest case: two goods in two markets, with linear demand and supply functions:

Good #1:

$$Q_{d1} - Q_{s1} = 0 \quad 2.18$$

$$Q_{d1} = a_0 + a_1 P_1 + a_2 P_2 \quad 2.19$$

$$Q_{s1} = b_0 + b_1 P_1 + b_2 P_2 \quad 2.20$$

Good #2:

$$Q_{d2} - Q_{s2} = 0 \quad 2.21$$

$$Q_{d2} = \alpha_0 + \alpha_1 P_1 + \alpha_2 P_2 \quad 2.22$$

$$Q_{s2} = \beta_0 + \beta_1 P_1 + \beta_2 P_2 \quad 2.23$$

Substitution 2.19 and 2.20 into 2.18, and 2.22 and 2.23 into 2.21, yields two equations in two unknown equilibrium prices:

$$(a_0 - b_0) + (a_1 - b_1)\bar{P}_1 + (a_2 - b_2)\bar{P}_2 = 0 \quad 2.24$$

$$(\alpha_0 - \beta_0) + (\alpha_1 - \beta_1)\bar{P}_1 + (\alpha_2 - \beta_2)\bar{P}_2 = 0 \quad 2.25$$

To simplify the notation, let  $c_i = a_i - b_i$ ,  $\gamma_i = \alpha_i - \beta_i$ , where  $i = 0,1,2$ . Equations 2.24 and 2.25 become

$$c_1 \bar{P}_1 + c_2 \bar{P}_2 = -c_0 \quad 2.26$$

$$\gamma_1 \bar{P}_1 + \gamma_2 \bar{P}_2 = -\gamma_0 \quad 2.27$$

Solving 2.26 for  $\bar{P}_1$  and substituting into 2.27 yields the equilibrium prices in terms of the model parameters alone:

$$\bar{P}_1 = \frac{c_2 \gamma_0 - c_0 \gamma_2}{c_1 \gamma_2 - c_2 \gamma_1} \quad 2.28$$

$$\bar{P}_2 = \frac{c_0\gamma_1 - c_1\gamma_0}{c_1\gamma_2 - c_2\gamma_1} \quad 2.29$$

We can use these equilibrium prices to find the equilibrium quantities from 2.18 through 2.23.

There are elegant proofs outlining the requirements for existence and uniqueness of the solutions to such systems; the matrix algebra proofs for more than two dimensions place conditions on the matrix determinants in the linear case, and on the jacobian determinants for the nonlinear case. In general, two things are demonstrated by these proofs: To yield a unique solution, the equations must be consistent (for example, nothing like  $x+y=8$ ,  $x+y=9$ ) and must be functionally independent (for example, nothing like  $x+y=2$ ,  $2x+2y=4$ ). Each equation must give information that is not contradicted by the others, and not already contained in the others.

## 8. Review of Calculus with Applications to Reduced-Form Comparative Statics

### a. Rules of Differentiation

One function, one variable:

1. Constant function: If  $y = f(x) = k$ , then

$$\frac{dy}{dx} = f'(x) = 0 \quad 2.30$$

2. Power function: If  $y = ax^n$ , then

$$f'(x) = nx^{n-1} \quad 2.31$$

Two functions, one variable:

3. Sum-Difference:  $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) = f'(x) \pm g'(x)$  2.32

4. Product:  $\frac{d}{dx}[f(x) \cdot g(x)] = f(x)g'(x) + g(x)f'(x)$  2.33

5. Quotient:  $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$  2.34

Quick Example: If  $f(x) = 30 + x^2 - 2x(4x + 3) + \frac{5x^3}{x^3 - 1}$ , find  $f'(x)$ .

By 2.32, the derivative of this sum is simply the sum of the derivatives of each term:

$$\frac{d}{dx}[30] = 0, \text{ by 2.30.}$$

$$\frac{d}{dx}[x^2] = 2x, \text{ by 2.31.}$$

$$\frac{d}{dx}[-2x(4x + 3)] = -2x(4) + (4x + 3)(-2), \text{ by 2.33.}$$

$$\text{Combining terms, } \frac{d}{dx}[-2x(4x + 3)] = -16x - 6.$$

$$\frac{d}{dx} \left[ \frac{5x^3}{x^3 - 1} \right] = \frac{(15x^2)(x^3 - 1) - (5x^3)(3x^2)}{(x^3 - 1)^2}, \text{ by 2.34.}$$

Combining terms,  $\frac{d}{dx} \left[ \frac{5x^3}{x^3 - 1} \right] = -15 \left[ \frac{x}{x^3 - 1} \right]$ .

Putting it all back together, we have

$$f'(x) = 2x - 16x - 6 - 15 \left[ \frac{x}{x^3 - 1} \right] = \frac{-14x^4 - 6x^3 - x + 6}{x^3 - 1}.$$

Two functions, two variables:

6. Chain:

If  $z = f(y)$  and  $y = g(x)$ , then

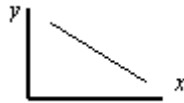
$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = f'(y)g'(x) \quad 2.35$$

7. Inverse Function Rule:

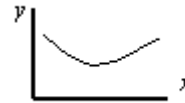
**Existence:** If  $y$  is a function of  $x$ ,  $y = f(x)$ , when may we also consider  $x$  to be a function of  $y$ ,  $x = f^{-1}(y)$ , read “ $x$  is an inverse function of  $y$ ?”

Answer: If and only if  $f(x)$  is monotonic, which we can check by finding the first derivative and checking to see that it does not change sign over the function’s domain. 2.36

Examples:



Yes



No

**Derivative of  $x = f^{-1}(y)$ :** If  $y = f(x)$ , and  $x = f^{-1}(y)$ , then

$$\frac{dx}{dy} = \frac{1}{dy/dx} \quad 2.37$$

Quick example: If  $y = x^9 + x^5 + 17$ , it’s difficult to find  $\frac{dx}{dy}$  by solving for  $x$ , but easy using the inverse

function rule:

$$\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{9x^8 + 5x^4}.$$

8. Partial differentiation: Given  $y = f(x_1, x_2, \dots, x_n)$ , where all  $x_i$  are independent,

then  $\frac{\partial y}{\partial x_i} = f_i(x)$ , read “the partial derivative of  $y$  with respect to  $x_i$ .” All other  $x_i$  are treated as constants when differentiating. 2.38

Quick example: Given  $R = P \cdot Q = P(a - bP + cA)$ , find  $\frac{\partial R}{\partial A}$ .

Using partial derivatives (assuming that  $P$  is not directly influenced by  $A$ ) and following the chain rule, we have  $\frac{\partial R}{\partial A} = Pc$ .

### 9. Logs and Exponential Functions

Though we will not see them again until Chapter 7, for completeness here are the laws for differentiating log and exponential functions:

$$\text{If } Y = \ln(X), \text{ then } \frac{dY}{dX} = \frac{1}{X}. \quad 2.39$$

$$\text{If } Y = e^X, \text{ then } \frac{dY}{dX} = e^X \text{ as well.} \quad 2.40$$

$$\text{If } Y = e^{a \cdot X}, \text{ then } \frac{dY}{dX} = a \cdot e^{a \cdot X} \quad 2.41$$

$$\text{If } Y = a^X, \text{ then } \frac{dY}{dX} = a^X \ln(a) \quad 2.42$$

### b. Applications to Reduced-Form Comparative Statics

Here we treat all parameters as independent. We “boil the system of equations down” to a “reduced form solution”—solutions for the endogenous variables that involve only explicit functions of the parameters and exogenous variables. Then we take partial derivatives of these reduced form equations, to find the “comparative static” effects of changes in parameters and exogenous variables.

**A simple microeconomic example:** Say we recall our earlier system, with an equilibrium condition:

$$Q_d = Q_s \quad 2.12$$

a demand curve:

$$Q_d = a - b \cdot P, \quad a, b > 0 \quad 2.13$$

and a supply curve:

$$Q_s = -c + d \cdot P \quad c, d > 0 \quad 2.14$$

This yielded the following reduced form solution:

$$\bar{P} = \frac{a + c}{b + d} \quad 2.16$$

$$\bar{Q} = \frac{ad - bc}{b + d} \quad 2.17$$

Now we can take partial derivatives to see how the equilibrium solution will change if the model’s parameters change:

$$\frac{\partial \bar{P}}{\partial a} = \frac{1}{b + d} > 0 \quad 2.43$$

In words: If the intercept of the demand curve increases, so will the equilibrium price.

We also have

$$\frac{\partial \bar{P}}{\partial c} = \frac{1}{b + d} > 0, \quad 2.44$$

and

$$\frac{\partial \bar{P}}{\partial b} = \frac{\partial \bar{P}}{\partial d} = \frac{-(a+c)}{(b+d)^2} < 0 \quad 2.45$$

**A simple microeconomic example:** Let

$$Y = C + I_0 + G_0 \quad 2.46$$

$$C = \alpha + \beta(Y - T), \quad \alpha > 0, \quad 0 < \beta < 1. \quad 2.47$$

$$T = \gamma + \delta Y, \quad \gamma > 0, \quad 0 < \delta < 1. \quad 2.48$$

This simple Keynesian model quietly assumes that

- aggregate supply passively responds to aggregate demand (the aggregate supply curve is horizontal),
- $I, G, \beta, \alpha, \gamma, \delta$  are all independent of each other; otherwise we would have “general functions” that can’t be boiled down to a simple reduced form. We’d have to either use a simulation, or use the implicit function rule to find total derivatives. This is beyond the scope of this course, but briefly: If

the functions have continuous, non-zero partial derivatives, then  $\frac{\partial y}{\partial x_i} = \frac{-F_i}{F_y}$ .

- Government spending and investment spending are exogenous, and taxes have both a fixed component (presumably taxes on investment) and a variable component (presumably on income, which is an endogenous variable).

Let’s find the equilibrium levels of two of the three endogenous variables: By substituting 2.48 into 2.47, then substituting the result into 2.46 and collecting terms, we have equilibrium national income,

$$\bar{Y} = \frac{\alpha - \beta\gamma + I_0 + G_0}{1 - \beta + \beta\delta}, \quad 2.49$$

and by inserting this result into 2.48, we get equilibrium tax revenues:

$$\bar{T} = \frac{\gamma + \delta \left[ \frac{\alpha + I_0 + G_0}{1 - \beta} \right]}{1 + \frac{\beta\delta}{1 - \beta}} \quad 2.50$$

Now for some interesting policy questions: How do increases in government spending affect equilibrium national income?

$$\frac{\partial \bar{Y}}{\partial G_0} = \frac{1}{1 - \beta + \beta\delta}, \text{ which is unambiguously greater than zero.} \quad 2.51$$

How about the effect upon national income when taxes on capital are increased?

$$\frac{\partial \bar{Y}}{\partial \gamma} = \frac{-\beta}{1 - \beta + \beta\delta}, \text{ which is unambiguously negative.} \quad 2.52$$

Can you find the effect upon national income when the marginal income tax rate,  $\delta$ , is increased? It’s

$$\frac{\partial \bar{Y}}{\partial \delta} = \frac{-\beta \bar{Y}}{1 - \beta + \beta\delta}, \text{ also unambiguously negative.} \quad 2.53$$

Taken together, those three results imply that government can increase GDP by increasing government spending, so long as taxes needn’t be increased. Why? Because in this model there is no money, and therefore no interest rate to be affected by government borrowing; investment spending is simply exogenous, determined outside our model.

Here are two short exercises to give you practice in finding comparative-static results:

- In our simple macro model, find  $\frac{\partial \bar{T}}{\partial \gamma}$ . (In other words, if capital taxes are lowered, is it possible that tax revenues actually increase—a supply-side utopia?)
- Replace the potentially-unrealistic assumption that government can increase spending without increasing taxes, yet without affecting investment spending: Force government to finance all spending with tax revenues, a balanced-budget amendment. That is, add a definition-equation that  $G_0$  must equal  $T$  from 2.48, which will allow you to completely remove  $G_0$  from the reduced-form equations. Now the only way to change  $G$  will be through changes in  $\gamma$  or  $\delta$ . Now recompute the comparative static results of 2.51 through 2.53, and interpret the differences that emerge.