RESOLUTIONS FOR METRIZABLE COMPACTA IN EXTENSION THEORY

Leonard R. Rubin and Philip J. Schapiro

Abstract. This is a summary of research which appears in a preprint of the same title. We prove a K-resolution theorem for simply connected CW-complexes K in extension theory in the class of metrizable compacta X. This means that if $\dim X \leq K$ (in the sense of extension theory), n is the first element of $\mathbb{N}$ such that $G = \pi_n(K) \neq 0$, and it is also true that $\pi_{n+1}(K) = 0$, then there exists a metrizable compactum Z and a surjective map $\pi : Z \to X$ such that:

(a) π is G-acyclic,
(b) $\dim Z \leq n + 1$, and
(c) $\dim Z \leq K$.

If additionally, $\pi_{n+2}(K) = 0$, then we may improve (a) to the statement,

(aa) π is K-acyclic.

To say that a map π is K-acyclic means that each map of each fiber $\pi^{-1}(x)$ to K is nullhomotopic.

In case $\pi_{n+1}(K) \neq 0$, we obtain a resolution theorem with a weaker outcome. Nevertheless, it implies the G-resolution theorem for arbitrary abelian groups G in cohomological dimension $\dim_G X \leq n$ when $n \geq 2$.

The Edwards-Walsh resolution theorem, the first resolution theorem for cohomological dimension, was proved in [Wa] (see also [Ed]). It states that if X is a metrizable compactum and $\dim_{\mathbb{Z}} X \leq n$ ($n \geq 0$), then there exists a metrizable compactum Z with $\dim Z \leq n$ and a surjective cell-like map $\pi : Z \to X$. This result, in conjunction with Dranishnikov's work ([Dr1]) showing that in the class of metrizable

1991 *Mathematics Subject Classification.* 55P55, 54F45.

Key words and phrases. G-acyclic resolution, K-acyclic resolution, dimension, cohomological dimension, cell-like map, shape of a point, inverse sequence, Edwards-Walsh resolution, simplicial complex, CW-complex, Moore space, Čech cohomology, Bockstein basis, Bockstein inequalities.

compacta, $\dim_{\mathbb{Z}}$ is distinct from dim, was a key ingredient for proving that cell-like maps could raise dimension (see [Ru1] for background). For the reader seeking fundamentals on the theory of cohomological dimension, $\dim_G$, the references [Ku], [Dr3], [Dy], and [Sh] could be helpful.

Now a map is cell-like provided that each of its fibers is cell-like, or, equivalently, has the shape of a point ([MS1]). Every cell-like compactum has trivial reduced Čech cohomology with respect to any abelian group G. This means that for every abelian group G, every cell-like map is G-acyclic, i.e., all its fibers have trivial reduced Čech cohomology with respect to the group G. This is equivalent to the statement that every map of such a fiber to $K(G,n)$ is nullhomotopic.

The latter notion may be generalized as follows. For a given CW-complex K, a metrizable compactum X is called *K-acyclic* if every map of it to K is nullhomotopic. Moreover, one should recall that when a Hausdorff compactum or metrizable space X has $\dim X \leq n$, then also $\dim_{\mathbb{Z}} X \leq n$.

With these ideas in mind, one may ask, what kind of parallel resolution theorems can be obtained under the assumption that $\dim_G X \leq n$, where G is an abelian group different from $\mathbb{Z}$? It turns out that it is not possible always to have cell-like resolutions as in the Edwards-Walsh theorem, nor can one even require in such propositions that $\dim Z \leq n$ be true (see [KY2]). So, what kind of resolution theorems can we expect? The main results of this paper go as follows.

1.1. Theorem. *Let K be a simply-connected CW-complex, n be the first element of $\mathbb{N}$ such that $G = \pi_n(K) \neq 0$, and X be a metrizable compactum with $\dim X \leq K$. Then there exists a metrizable compactum Z and a surjective map $\pi : Z \to X$ such that:*

(a) *π is G-acyclic,*
(b) *$\dim Z \leq n+1$, and*
(c) *$\dim_G Z \leq n$.*

1.2. Theorem. *Let K be a simply-connected CW-complex, n be the first element of $\mathbb{N}$ such that $G = \pi_n(K) \neq 0$, and assume that $\pi_{n+1}(K) = 0$. Then for each metrizable compactum X with $\dim X \leq K$, there exists a metrizable compactum Z and a surjective map $\pi : Z \to X$ such that:*

(a) *π is G-acyclic,*
(b) *$\dim Z \leq n+1$, and*
(c) *$\dim Z \leq K$.*

If in addition, $\pi_{n+2}(K) = 0$, then we may also conclude that

(aa) *π is K-acyclic.*

If $K = K(G, n)$, then $\dim X \leq K$ is equivalent to $\dim_G X \leq n$. Hence, as a corollary to Theorem 1.1, we get the G-acyclic resolution theorem in cohomological dimension theory.

1.3. Corollary. *Let G be an abelian group and X be a metrizable compactum with $\dim_G X \leq n$ ($n \geq 2$). Then there exists a metrizable compactum Z and a surjective map $\pi : Z \to X$ such that:*

(a) *π is G-acyclic,*
(b) $\dim Z \leq n + 1$, *and*
(c) $\dim_G Z \leq n$.

In [Le] one finds another approach to 1.3. We mention that the Edwards-Walsh theorem has been generalized to the class of arbitrary metrizable spaces by Rubin and Schapiro ([RS]) and to the class of arbitrary compact Hausdorff spaces by Mardešić and Rubin ([MR]). Corollary 1.3 was proved by Dranishnikov ([Dr2]) for the group $G = \mathbb{Z}/p$, where p is an arbitrary prime number, but with the stronger outcome that $\dim Z \leq n$. Later, Koyama and Yokoi ([KY1]) were able to obtain this $\mathbb{Z}/p$-resolution theorem of Dranishnikov both for the class of metrizable spaces and for that of compact Hausdorff spaces.

In their work [KY2], Koyama and Yokoi have made a substantial amount of progress in the resolution theory of metrizable compacta, that is, towards proving Corollary 1.3. Their method relies heavily on the existence of Edwards-Walsh resolutions, which had been studied by Dydak and Walsh in [DW], and which had been applied originally, in a rudimentary form, in [Wa]. The definition of an Edwards-Walsh resolution can be found in [KY2], but we shall not use it herein.

To overcome a flaw in the proof of Lemma 4.4 of [DW], Koyama and Yokoi proved the existence of Edwards-Walsh resolutions for some groups G, but under a stronger set of assumptions on G than had been thought necessary in [DW]. It is still not known if these stronger assumptions are needed to insure the existence of the resolutions. Nevertheless, Koyama and Yokoi were able to prove substantial G-acyclic resolution theorems. Let us state two of the important theorems from [KY2] (Theorems 4.9 and 4.12, respectively), which greatly influenced the direction of the work in this paper.

1.4. Theorem. *Corollary 1.3 is true for every torsion free abelian group G.*

1.5. Theorem. *Let G be an arbitrary abelian group and X be a metrizable compactum with $\dim_G X \leq n$, $n \geq 2$. Then there exists a surjective G-acyclic map $\pi : Z \to X$ from a metrizable compactum Z where $\dim Z \leq n+2$ and $\dim_G Z \leq n+1$.*

In case G is a torsion group, they prove (Theorem 4.11) that Corollary 1.3 holds, but without part (c). Of course Theorem 1.5 falls short of Corollary 1.3. We observed that one of the main reasons for the relative weakness of this theorem was that Koyama and Yokoi proved it by an indirect technique, a type of "finesse." Their approach depends heavily on the Bockstein basis theorem and the Bockstein inequalities (see [Ku]), instead of the more direct method, involving Edwards-Walsh resolutions, used to prove Theorem 1.4.

We want to point out that Theorem 1.4 includes as a corollary, and therefore redeems, the $\mathbb{Q}$-resolution theorem of Dranishnikov ([Dr5]–but see also [Dr6] where a different proof is given). The Koyama and Yokoi proof shows that in the proof of Theorem 3.2 of [Dr5], the statement that $\alpha_m \circ \omega_m$ is an Edwards-Walsh resolution over $\tau_m^{(n+1)}$ is not true. This was a subtle point; to fully understand it, the interested reader may examine the text immediately following the proof of Fact 1 of the proof of Theorem 3.1 in [KY2]. Getting around the barrier naturally led to a quite complicated construction.

Our proof of Theorems 1.1 and 1.2 will be direct, using extensions which are different from Edwards-Walsh resolutions. But we will use a type of pseudo-Bockstein basis denoted $\sigma_0(G)$ (section 9). This will allow us to deal with the groups $\mathbb{Z}/p^\infty$ as well as the other groups involved. We shall employ the technique of inverse sequences both to represent our given space X and to determine the resolving space Z. The map $\pi : Z \to X$ will be obtained in a standard, yet complicated manner similar to that used in [Wa].

References

[Dr1] A. Dranishnikov, *On P. S. Aleksandrov's problem*, Mat. Sb. **135 (4)** (1988), 551–557.

[Dr2] A. Dranishnikov, *On homological dimension modulo p*, Math. USSR Sbornik **60, no. 2** (1988), 413–425.

[Dr3] A. Dranishnikov, *Homological dimension theory*, Russian Math. Surveys **43(4)** (1988), 11–63.

[Dr4] A. Dranishnikov, *K-theory of Eilenberg-MacLane spaces and cell-like mapping problem*, Trans. Amer. Math. Soc. **335:1** (1993), 91–103.

[Dr5] A. Dranishnikov, *Rational homology manifolds and rational resolutions*, Topology and its Appls. **94** (1999), 75–86.

[Dr6] A. Dranishnikov, *Cohomological dimension theory of compact metric spaces*, preprint.

[DD] A. Dranishnikov and J. Dydak, *Extension dimension and extension types*, Tr. Mat. Inst. Steklova **212** (1996), 61–94.

[Dy] J. Dydak, *Handbook of Geometric Topology*, Elsevier, Amsterdam, 2002, pp. 423–470.

[DW] J. Dydak and J. Walsh, *Complexes that arise in cohomological dimension theory: a unified approach*, J. London Math. Soc. **(2) (48), no. 2** (1993), 329–347.

[Ed] R. D. Edwards, *A theorem and question related to cohomological dimension and cell-like maps*, Notices of the Amer. Math. Soc. **25** (1978), A-259.

[Hu] S. Hu, *Homotopy Theory*, Academic Press, New York, 1959.

[JR] R. Jimenez and L. Rubin, *An addition theorem for n-fundamental dimension in metric compacta*, Topology and its Appls. **62** (1995), 281–297.

[KY1] A. Koyama and K. Yokoi, *A unified approach of characterizations and resolutions for cohomological dimension modulo p*, Tsukuba J. Math. **18(2)** (1994), 247–282.

[KY2] A. Koyama and K. Yokoi, *Cohomological dimension and acyclic resolutions*, Topology and its Appls., to appear.

[Ku] W. I. Kuz'minov, *Homological Dimension Theory*, Russian Math. Surveys **23** (1968), 1–45.

[Le] M. Levin, *Acyclic resolutions for arbitrary groups*, preprint.

[Ma] S. Mardešić, *Extension dimension of inverse limits*, Glasnik Mat. **35(55)** (2000), 339–354.

[MR] S. Mardešić and L. Rubin, *Cell-like mappings and nonmetrizable compacta of finite cohomological dimension*, Trans. Amer. Math. Soc. **313** (1989), 53–79.

[MS1] S. Mardešić and J. Segal, *Shape Theory*, North-Holland, Amsterdam, 1982.

[MS2] S. Mardešić and J. Segal, *Stability of almost commutative inverse systems of compacta*, Topology and its Appls. **31** (1989), 285–299.

[Ru1] L. Rubin, *Cell-like maps, dimension and cohomological dimension: a survey*, Banach Center Publications **18** (1986), 371–376.

[Ru2] L. Rubin, *Cohomological dimension and approximate limits*, Proc. Amer. Math. Soc. **125** (1997), 3125–3128.

[Ru3] L. Rubin, *A stronger limit theorem in extension theory*, Glasnik Mat. **36(56)** (2001), 95–103.

[RS] L. Rubin and P. Schapiro, *Cell-like maps onto non-compact spaces of finite cohomological dimension*, Topology and its Appls. **27** (1987), 221–244.

[Sh] E. Shchepin, *Arithmetic of dimension theory*, Russian Math. Surv. **53** (1998), 975–1069.

[Sp] E. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.

[Wa] J. Walsh, *Shape Theory and Geometric Topology*, Lecture Notes in Mathematics, volume 870, Springer Verlag, Berlin, 1981, pp. 105–118.

Department of Mathematics, University of Oklahoma, 601 Elm Ave., Rm. 423, Norman, OK 73019 USA

E-mail address: lrubin@ou.edu

Department of Mathematics, Langston University, Langston, OK 73050 USA

E-mail address: pjschapiro@lunet.edu