ON HOMOTOPY PROPERTIES OF CERTAIN COXETER GROUP BOUNDARIES

HANSPETER FISCHER AND CRAIG R. GUILBAULT

Abstract. There is a canonical homomorphism $\psi : \pi_1(\text{bdy } X) \to \pi_1^\infty(X)$ from the fundamental group of the visual boundary, here denoted by bdy X, of any non-positively curved geodesic space X into its fundamental group at infinity. In this setting, the latter group coincides with the first shape homotopy group of the visual boundary: $\pi_1^\infty(X) \equiv \check{\pi}_1(\text{bdy } X)$. The induced homomorphism $\varphi : \pi_1(\text{bdy } X) \to \check{\pi}_1(\text{bdy } X)$ provides a way to study the relationship between these groups.

We present a class $\mathcal{Z}$ of compacta, so-called *trees of manifolds*, for which we can show that the homomorphisms $\varphi : \pi_1(Z) \to \check{\pi}_1(Z)$ ($Z \in \mathcal{Z}$) are injective. This class $\mathcal{Z}$ includes the visual boundaries $Z = \text{bdy } X$ which arise from right-angled Coxeter groups whose nerves are closed PL-manifolds. In particular, it includes the visual boundaries of those Coxeter groups which act on Davis' exotic open contractible manifolds [2].

1. The first shape homotopy group of a metric compactum

We recall the definition of the first shape homotopy group of a pointed compact metric space (Z, z_0). Choose an inverse sequence

$$(Z_1, z_1) \xleftarrow{f_{2,1}} (Z_2, z_2) \xleftarrow{f_{3,2}} (Z_3, z_3) \xleftarrow{f_{4,3}} \cdots$$

of pointed compact polyhedra such that

$$(Z, z_0) = \varprojlim((Z_i, z_i), f_{i+1,i}).$$

The *first shape homotopy group* of Z based at z_0 is then given by

$$\check{\pi}_1(Z, z_0) = \varprojlim \left(\pi_1(Z_1, z_1) \xleftarrow{f_{2,1\#}} \pi_1(Z_2, z_2) \xleftarrow{f_{3,2\#}} \pi_1(Z_3, z_3) \xleftarrow{f_{4,3\#}} \cdots\right).$$

This definition of $\check{\pi}_1(Z, z_0)$ does not depend on the choice of the sequence
$((Z_i, z_i), f_{i+1,i})$ [8]. Let $p_i : (Z, z_0) \to (Z_i, z_i)$ be the projections of the limit (Z, z_0) into its inverse sequence $((Z_i, z_i), f_{i+1,i})$ such that $p_i = f_{i+1,i} \circ p_{i+1}$ for all i. Since the maps p_i induce homomorphisms

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$p_{i\#} : \pi_1(Z, z_0) \to \pi_1(Z_i, z_i)$ such that $p_{i\#} = f_{i+1,i\#} \circ p_{i+1\#}$ for all i, we obtain an induced homomorphism $\varphi : \pi_1(Z, z_0) \to \check{\pi}_1(Z, z_0)$ given by $\varphi([\alpha]) = ([\alpha_1], [\alpha_2], [\alpha_3], \cdots)$, where $\alpha_i = p_i \circ \alpha$.

The following examples illustrate that $\varphi : \pi_1(Z, z_0) \to \check{\pi}_1(Z, z_0)$ need not be injective and is typically not surjective.

Example 1. Let

$Y = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0, 0 < x \leq 1, y = \sin 1/x\} \cup (\{0\} \times [-1, 1] \times \{0\})$

be the "topologist's sine curve". Define $Y_i = Y \cup ([0, 1/i] \times [-1, 1] \times \{0\})$. Let Z and Z_i be the subsets of $\mathbb{R}^3$ obtained by revolving Y and Y_i about the y-axis, respectively, and let $f_{i+1,i} : Z_{i+1} \hookrightarrow Z_i$ be inclusion. Then Z is the limit of the inverse sequence $(Z_i, f_{i+1,i})$. If we take $z_0 = (1, \sin 1, 0)$, then $\pi_1(Z, z_0)$ is infinite cyclic, while $\check{\pi}_1(Z, z_0)$ is trivial.

Example 2. We can make the space Z of the previous example path connected, by taking any arc $a \subseteq \mathbb{R}^3$, such that $a \cap Z = \partial a = \{z_0, (0, 1, 0)\}$, and then considering $Z^+ = Z \cup a$. Notice that both $\pi_1(Z^+, z_0)$ and $\check{\pi}_1(Z^+, z_0)$ are infinite cyclic. However, the homomorphism $\varphi : \pi_1(Z^+, z_0) \to \check{\pi}_1(Z^+, z_0)$ is trivial.

Example 3. Let $Z = \bigcup_{k=1}^{\infty} C_k$ be the Hawaiian Earrings, where

$$C_k = \{(x, y) \in \mathbb{R}^2 \mid x^2 + (y - 1/k)^2 = (1/k)^2\}.$$

Put $Z_i = C_1 \cup C_2 \cup \cdots \cup C_i$ and let $z_0 = z_i = (0, 0)$. Define $f_{i+1,i} : Z_{i+1} \to Z_i$ by $f_{i+1,i}(p) = (0, 0)$ for $p \in C_{i+1}$ and $f_{i+1,i}(p) = p$ for $p \in Z_{i+1} \setminus C_{i+1}$. Then (Z, z_0) is the limit of the inverse sequence $((Z_i, z_i), f_{i+1,i})$. While this time $\varphi : \pi_1(Z, z_0) \to \check{\pi}_1(Z, z_0)$ is injective [4], it is not surjective: let $l_i : (S^1, *) \to (C_i, z_0)$ be a fixed homeomorphism and consider for each i the element

$g_i = [l_1][l_1][l_1]^{-1}[l_1]^{-1}[l_1][l_2][l_1]^{-1}[l_2]^{-1}[l_1][l_3][l_1]^{-1}[l_3]^{-1} \cdots [l_1][l_i][l_1]^{-1}[l_i]^{-1}$

of $\pi_1(Z_i, z_i)$. Then the sequence $(g_i)_i$ is an element of the group $\check{\pi}_1(Z, z_0)$ which is clearly not in the image of φ.

2. Trees of manifolds

We shall call a topological space Z a *tree of manifolds* if there is an inverse sequence

$$M_1 \xleftarrow{f_{2,1}} M_2 \xleftarrow{f_{3,2}} M_3 \xleftarrow{f_{4,3}} \cdots,$$

called a *defining sequence* for Z, of distinct closed PL-manifolds M_n with collared disks $D_n \subseteq M_n$, and continuous functions $f_{n+1,n} : M_{n+1} \to M_n$ that have the following properties:

(P-1) $$Z = \varprojlim \left(M_1 \xleftarrow{f_{2,1}} M_2 \xleftarrow{f_{3,2}} M_3 \xleftarrow{f_{4,3}} \cdots \right);$$

(P-2) For each n, the restriction of $f_{n+1,n}$ to the set $f_{n+1,n}^{-1}(M_n \setminus \text{int } D_n)$, call it $h_{n+1,n}$, is a homeomorphism onto $M_n \setminus \text{int } D_n$, and $h_{n+1,n}^{-1}(\partial D_n)$ is bicollared in M_{n+1};

(P-3) For each n, $\lim_{m\to\infty} \text{diam } [f_{m,n}(D_m)] = 0$, where $f_{m,n} = f_{n+1,n} \circ f_{n+2,n+1} \circ \cdots \circ f_{m,m-1} : M_m \to M_n$ & $f_{n,n} = id_{M_n}$.

(P-4) For each pair $n < m$, $f_{m,n}(D_m) \cap \partial D_n = \emptyset$.

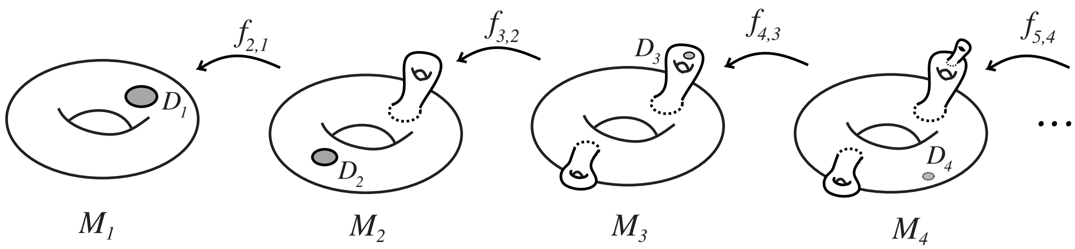

FIGURE 1. A tree of manifolds

It follows that, for $m \geq n+2$, the set

$$E_{m,n} = \text{int } D_n \cup f_{n+1,n}(\text{int } D_{n+1}) \cup f_{n+2,n}(\text{int } D_{n+2}) \cup \cdots \cup f_{m-1,n}(\text{int } D_{m-1})$$

can be written as the union of $m-n$, or fewer, open disks in M_n and that $f_{m,n}$ restricted to $f_{m,n}^{-1}(M_n \setminus E_{m,n})$ is a homeomorphism onto $M_n \setminus E_{m,n}$, which we will denote by $h_{m,n}$. Moreover, if for $n < m$ we define the spheres $S_{m,n} = h_{m,n}^{-1}(\partial D_n) \subseteq M_m$, we see that the collection $\mathcal{S}_n = \{S_{n,1}, S_{n,2}, \cdots, S_{n,n-1}\}$ decomposes M_n into a connected sum

$$M_n = [N_{n,1} \# N_{n,2} \# \cdots \# N_{n,n-1}] \# N_{n,n} \approx M_{n-1} \# N_{n,n}.$$

Hence, Z can be thought of as the limit of a growing tree of connected sums of closed manifolds. In particular, in dimensions greater than two, we have

$$\pi_1(M_n) = \pi_1(N_{n,1}) * \pi_1(N_{n,2}) * \cdots * \pi_1(N_{n,n-1}) * \pi_1(N_{n,n});$$

and in dimension two, we have

$$\pi_1(M_n) = F_{n,1} *_{\pi_1(S_{n,1})} F_{n,2} *_{\pi_1(S_{n,2})} \cdots *_{\pi_1(S_{n,n-2})} F_{n,n-1} *_{\pi_1(S_{n,n-1})} F_{n,n},$$

where $F_{n,i}$ denotes the free fundamental group of the appropriately punctured $N_{n,i}$.

Note also that each $S_{n,i} \approx \partial D_i$ naturally embeds in Z.

Definition. We will call a defining sequence $M_1 \xleftarrow{f_{2,1}} M_2 \xleftarrow{f_{3,2}} M_3 \xleftarrow{f_{4,3}} \cdots$ *well-balanced* if the set $\bigcup_{m\geq 3} E_{m,1}$ either has finitely many components or is dense in M_1, and if for each $n \geq 2$, the set $h_{n,n-1}^{-1}(M_{n-1} \setminus$

$D_{n-1}) \cup \left[\bigcup_{m \geq n+2} E_{m,n} \right]$ either has finitely many components or is dense in M_n.

Whether Z has a well-balanced defining sequence or not, will play a role only in the case when the manifolds M_n are 2-dimensional closed surfaces. Specifically, our main result is the following

Theorem. *Suppose Z is a tree of manifolds, and $z_0 \in Z$. In case Z is 2-dimensional, suppose further that Z admits a well-balanced defining sequence. Then the canonical homomorphism $\varphi : \pi_1(Z, z_0) \to \check{\pi}_1(Z, z_0)$ is injective.*

Remark. In case $\pi_1(N_{n,n}) \neq 1$ for infinitely many n, an argument analogous to Example 3 shows that $\varphi : \pi_1(Z, z_0) \to \check{\pi}_1(Z, z_0)$ is not surjective.

For a detailed proof of this theorem see [6]. Here, we only give a brief

SKETCH OF PROOF. Since it is known that the canonical homomorphism
$\pi_1(Y) \to \check{\pi}_1(Y)$ is injective for all 1-dimensional compacta Y [4], we will assume that $\dim Z \geq 2$.

Let $\alpha : S^1 \to Z$ be a loop such that $\alpha_n = p_n \circ \alpha : S^1 \to M_n$ is nullhomotopic for each n. We wish to show that $\alpha : S^1 \to Z$ is nullhomotopic. We will do this by constructing a map $\beta : D^2 \to Z$ with $\beta|_{S^1} = \alpha$. By assumption, we may choose maps $\beta_n : D^2 \to M_n$ with $\beta_n|_{S^1} = \alpha_n$. The difficulty of the proof, of course, is that in general $\beta_n \neq f_{n+1,n} \circ \beta_{n+1}$, so that the sequence $(\beta_n)_n$ does not even constitute a function $D^2 \to Z$ into the inverse limit, let alone a map extending α.

Although we might not be in a position to move the maps α_n the slightest bit, we can place β_n in general position with respect to the spheres of the collection $\mathcal{S}_n$ while having $\beta_n|_{S^1}$ approximate α_n with increasing accuracy as n increases. Indeed, we can arrange for each *cancellation pattern* $\beta_n^{-1}(\bigcup \mathcal{S}_n)$ to consist of finitely many pairwise disjoint straight line segments in D^2 which have their endpoints in S^1. Ideally, we would like to paste together our map β from appropriate pieces belonging to the maps of the sequence $(\beta_n)_n$, namely those pieces that cancel the elements of $\pi_1(N_{n,n})$. However, these cancellation patterns will in general not be compatible. For example, in dimensions greater than two, the cancellation pattern for an element

$$[\alpha_{n+1}] = h_1 * k_1 * h_2 * k_2 * \cdots * h_5 * k_5 = 1 \in \pi_1(M_{n+1}) = \pi_1(M_n) * \pi_1(N_{n+1,n+1})$$

might be witnessed by β_{n+1} as

$$(h_1(k_1(h_2(k_2)h_3)k_3(h_4)k_4)h_5(k_5)) = 1.$$

The induced cancellation pattern for

$$[\alpha_n] = f_{n+1,n\#}([\alpha_{n+1}]) = h_1 * 1 * h_2 * 1 * \cdots * 1 * h_5 * 1 = 1 \in \pi_1(M_n) * \{1\}$$

as obtained from $f_{n+1,n} \circ \beta_{n+1}$ would then be given by

$$(h_1((h_2h_3)(h_4))h_5) = 1.$$

On the other hand, the map β_n might cancel $[\alpha_n]$ as

$$((h_1h_2)(h_3(h_4)h_5)) = 1.$$

This is illustrated in Figure 2, which depicts the sets $\beta_n^{-1}(\partial D_n)$, $(f_{n+1,n} \circ \beta_{n+1})^{-1}(\partial D_n)$, and $\beta_{n+1}^{-1}(S_{n+1,n})$ as dashed lines.

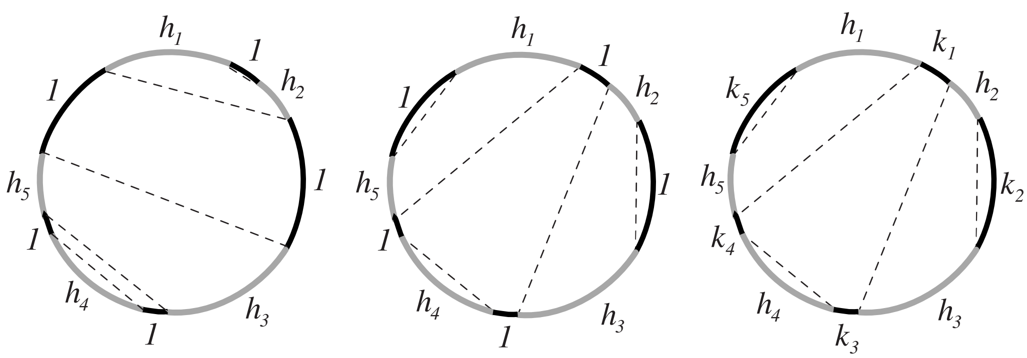

FIGURE 2.

If k_1 is not trivial and if k_3 does not cancel k_4 in $\pi_1(N_{n+1,n+1})$, then we cannot use any of the pieces of the map β_n to construct β.

As a remedy, we repeatedly select subsequences until, at least approximately, all cancellation patterns are coherent. That is, until the sets $\beta_n^{-1}(\bigcup S_n)$ are approximately nested with increasing n. Once this is achieved, the union of these cancellation patterns produce a limiting pattern $\mathcal{P}$ of possibly infinitely many straight line segments in D^2 whose interiors are pairwise disjoint and whose endpoints lie in S^1. Each segment of $\mathcal{P}$, at least approximately, maps under some β_n into some $S_{n,i}$. Note that we must allow for the possibility that α_n meets some $S_{n,i}$ in infinitely many points. This effect is accounted for by a possible increase of segments $c \subseteq \beta_m^{-1}(\bigcup \mathcal{S}_m)$ for which $\beta_m(c) \subseteq S_{m,i}$, as m increases. The map $\beta : D^2 \to Z$ can now be defined in two stages.

First, extend $\alpha : S^1 \to Z$ to a map $\beta : S^1 \cup \mathcal{P} \to Z$. If dim $Z = 2$, this can be done so that each segment of $\mathcal{P}$ maps to a local geodesic of that simple closed curve of Z which corresponds to the appropriate

∂D_i. If dim $Z \geq 3$, any coherent extension into the spheres of Z corresponding to ∂D_i will do, so long as the extension to a segment does not deviate too much from the image of its endpoints. If all this is done with sufficient care, the map $\beta : S^1 \cup \mathcal{P} \to Z$ will be uniformly continuous, so that we can extend it to the closure of its domain.

Next, focus on the components of the subset of D^2 on which the map β is not yet defined. Call these components *holes*. The boundary, bdy H, of a hole H is a simple closed curve, which either maps to a singleton under β, in which case we extend β trivially over cl H, or $p_n \circ \beta(\text{bdy } H) \subseteq N_n^*$ for some n, where

$$N_1^* = M_1 \setminus \Big(\bigcup_{m \geq 3} E_{m,1} \Big)$$

and

$$N_n^* = M_n \setminus \Big[h_{n,n-1}^{-1}(M_{n-1} \setminus D_{n-1}) \cup \Big(\bigcup_{m \geq n+2} E_{m,n} \Big) \Big] \text{ for } n \geq 2.$$

The map $p_n \circ \beta$: bdy $H \to N_n^* \subseteq M_n$ can be extended to a map $p_n \circ \beta$: cl $H \to M_n$ so long as the hole H is sufficiently "thin", because M_n is an ANR. For the moment, assume that dim $Z \geq 3$. The map $p_n \circ \beta$: cl $H \to M_n$ can then be cut off at $S_{n,n-1} = h_{n,n-1}^{-1}(\partial D_{n-1})$ and pushed off $\bigcup_{m \geq n+2} E_{m,n}$. Hence, we may extend the map $p_n \circ \beta$: bdy $H \to N_n^*$ to a map $p_n \circ \beta$: cl $H \to N_n^*$. Since N_n^* naturally embeds in Z, we have an extension of β : bdy $H \to Z$ to β : cl $H \to Z$. For each n, there will be *finitely* many maps $p_n \circ \beta$: bdy $H \to N_n^* \subseteq M_n$ for which the hole H is not thin enough to make this argument. In those cases, some $f_{m,n} \circ \beta_m : D^2 \to M_n$, with sufficiently large m, will be witness to the fact that $p_n \circ \beta$: bdy $H \to M_n$ is nullhomotopic after all. This is due to the approximate nestedness of the cancellation patterns $\beta_n^{-1}(\bigcup \mathcal{S}_n)$. Since for sufficiently large n the subset of Z which is homeomorphic to N_n^* is arbitrarily small, this procedure guarantees continuity of the resulting map $\beta : D^2 \to Z$.

If dim $Z = 2$, the above process requires a little bit more care and is helped by the assumption that the defining tree is well-balanced. Specifically, the sets N_n^* will either be ANRs or 1-dimensional. In the former case, we can adapt the argument we just made, and in the latter case, we make use of the result in [4] mentioned at the beginning of this proof. □

3. An application to Coxeter group boundaries

We now present an application of our theorem to boundaries of certain non-positively curved geodesic spaces. Recall that a metric space is *proper* if all of its closed metric balls are compact. A *geodesic space* is a metric space in which any two points lie in a geodesic, i.e. a subset that is isometric to an interval of the real line in its usual metric. A proper geodesic space X is said to be *non-positively curved* if any two points on the sides of a geodesic triangle in X are no further apart than their corresponding points on a reference triangle in Euclidean 2-space. The *visual boundary* of a non-positively curved geodesic space X, denoted by bdy X, is defined to be the set of all geodesic rays emanating from a fixed point x_0 endowed with the compact open topology. Let some geodesic base-ray $\omega : [0,\infty) \to X$ with $\omega(0) = x_0$ be given. Under the relatively mild assumption that the pointed concentric metric spheres $(S_{x_0}(i), \omega(i))$ have the pointed homotopy type of ANRs, it is shown in [1], that

$$\check{\pi}_1(\text{bdy } X, \omega) = \pi_1^\infty(X, \omega).$$

Here, $\pi_1^\infty(X,\omega)$ is the *fundamental group at infinity* of X, that is, the limit of the sequence

$$\pi_1(X \setminus B(1), \omega(2)) \leftarrow \pi_1(X \setminus B(2), \omega(3)) \leftarrow \pi_1(X \setminus B(3), \omega(4)) \leftarrow \cdots$$

whose bonds are induced by inclusion followed by a base point slide along ω.

A class of visual boundaries to which our theorem applies, arises from non-positively curved simplicial complexes, which are acted upon by certain Coxeter groups, whose definition we now briefly recall: let V be a finite set and $m : V \times V \to \{\infty\} \cup \{1,2,3,\cdots\}$ a function with the property that $m(u,v) = 1$ if and only if $u = v$, and $m(u,v) = m(v,u)$ for all $u,v \in V$. Then the group $\Gamma = \langle V \mid (uv)^{m(u,v)} = 1 \text{ for all } u,v \in V \rangle$ defined in terms of generators and relations is called a *Coxeter group*. If moreover $m(u,v) \in \{\infty, 1, 2\}$ for all $u,v \in V$, then Γ is called *right-angled*. The abstract simplicial complex $N(\Gamma, V) = \{\emptyset \neq S \subseteq V \mid S \text{ generates a finite subgroup of } \Gamma\}$ is called the *nerve* of the group Γ. For a right-angled Coxeter group, the isomorphism type of the nerve $N(\Gamma, V) = N(\Gamma)$ does not depend on the Coxeter system (Γ, V) but only on the group Γ [10].

For the remainder of this discussion, let Γ be a right-angled Coxeter group whose nerve $N(\Gamma)$ is a closed PL-manifold. This includes, for example, the Coxeter groups generated by the reflections of any one of Davis' exotic open contractible n-manifolds ($n \geq 4$), for which the nerves are PL-homology spheres [2].

As described, for example, in [3], Γ acts properly discontinuously on a non-positively curved (and hence contractible) simplicial complex $X(\Gamma)$, its so-called *Davis-Vinberg complex*, by isometry and with compact quotient. In [5] it is shown that the visual boundary of $X(\Gamma)$ is a (well-balanced) tree of manifolds. (By virtue of [11], the proof given in [5] also applies to the non-orientable case.) The visual boundary of $X(\Gamma)$ is usually referred to as the *boundary* of Γ and is denoted by bdy Γ. Since Coxeter groups are semi-stable at infinity [9] and Γ is one-ended, $\pi_1^\infty(X(\Gamma), \omega) = \pi_1^\infty(\Gamma)$ is actually an invariant of the group Γ [7].

In summary, we obtain the following

Corollary. *Let Γ be a right-angled Coxeter group whose nerve $N(\Gamma)$ is a closed PL-manifold. Then the canonical homomorphism $\psi : \pi_1(bdy\ \Gamma) \to \pi_1^\infty(\Gamma)$ is injective.*

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Department of Mathematical Sciences, Ball State University, Muncie, IN 47306 USA

E-mail address: `fischer@math.bsu.edu`

Department of Mathematical Sciences, University of Wisconsin–Milwaukee, Milwaukee, WI 53201 USA

E-mail address: `craigg@uwm.edu`