DIMENSION THEORY LOCAL AND GLOBAL

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Dedicated to Bob Daverman on the occasion of his 60th birthday.

These survey lectures are devoted to a new subject of the large scale dimension theory which was initiated by Gromov as a part of asymptotic geometry. We are going to enter the large scale world and consider some new concepts, results and examples which are parallel in many cases to the corresponding elements of the standard (local) dimension theory. We start our presentation with the motivations.

Lecture 1. MOTIVATONS and CONCEPTS

1.1. **Big picture of the Novikov Conjecture.** The Novikov Conjecture (NC) states that *the higher signatures of a manifold are homotopy invariant.* The *higher signatures* are the rational numbers of the type $\langle L(M) \cup \rho_M^*(x), [M]\rangle$, where $[M]$ is the fundamental class of a manifold M, L is the Hirzebruch class, $\Gamma = \pi_1(M)$, $\rho_M : M \to B\Gamma = K(\Gamma, 1)$ is a map classifying the universal cover of M and $x \in H^*(B\Gamma; \mathbb{Q})$ is a rational cohomology class. The name 'higher signature' is due to the Hirzebruch signature formula $\sigma(M) = \langle L(M), [M]\rangle$. It is known that the higher signatures are the only possible homotopy invariant characteristic numbers. It is convenient to formulate the NC for groups Γ instead of manifolds. We say that *the Novikov Conjecture holds for a discrete group* Γ *if it holds for all manifolds* M *(closed, orientable) with the fundamental group* $\pi_1(M) = \Gamma$. One of the reason for this is that the conjecture is verified for many large classes of groups. The other reason is that the Novikov Conjecture for the group can be reformulated in terms of the surgery exact sequence: *The rational Wall assembly map*

$$l_*^\Gamma : H_*(B\Gamma; \mathbb{Q}) \to L_*(\pi) \otimes \mathbb{Q}$$

is a monomorphism [Wa], [FRR],[KM].

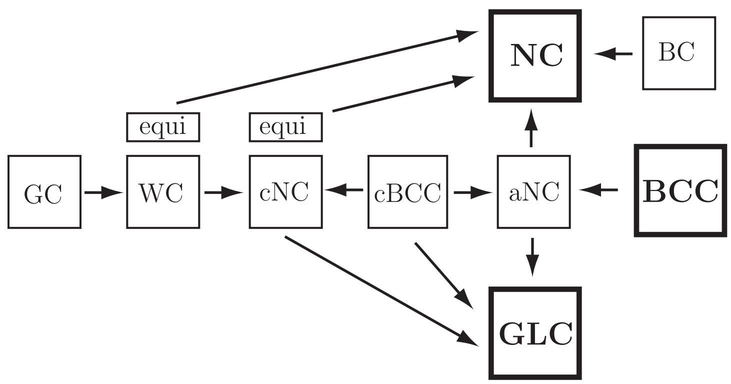
1991 *Mathematics Subject Classification.* Primary 53C23, 57R22, 57S30, 20F65, 05C10.

Key words and phrases. hypereuclidean manifold, expander, aspherical manifold, Novikov conjecture.

The author was partially supported by NSF grant DMS-9971709.

The case when $B\Gamma$ is a finite complex is of a particular interest here. In this case the group Γ taken as a metric space in the word metric is coarsely equivalent to the universal cover $E\Gamma$. This makes the methods of asymptotic geometry more natural. According to Davis' trick every finite aspherical complex K is a retract of a closed aspherical orientable manifold M [D]. Then the diagram formed by the surgery exact sequence and this retraction implies that if the NC holds for $\pi_1(M)$, then it holds for $\pi_1(K)$. Having that in mind from this moment we will stick to the case when $B\Gamma = M$ is a closed orientable manifold. Since M is aspherical, without loss of generality we may assume that the universal cover X of M is homeomorphic to a euclidean space.

There are several famous conjectures about aspherical manifolds. We arrange them in the following picture.



Here we assume that Γ is fixed and $B\Gamma = M$ is a closed manifold of dimension n. We note that almost all these conjectures are stated in more general form. With the above restriction they form this picture where every arrow is a theorem.

Below we give a brief description of the conjectures.

Borel Conjecture (BC). *Every homotopy equivalence between closed aspherical manifolds is homotopic to a homeomorphism.*

The arrow BC $\to$ NC follows from the surgery exact sequence [FRR].

Gromov-Lawson Conjecture (GLC). *A closed aspherical manifold cannot carry a metric of positive scalar curvature.*

The scalar curvature of an n-dimensional Riemannian manifold M at a point x can be defined up to a constant multiple as

$$\lim_{r\to 0} \frac{Vol\ B_r(\mathbb{R}^n, 0) - Vol\ B_r(M, x)}{r^{n+2}},$$

where $B_r(X, x)$ denotes the r-ball in a metric space X centered at x.

Analytic Novikov Conjecture (aNC). *The analytic assembly map* $\mu : K_*(B\Gamma) \to K_*(C_r^*(\Gamma))$ *is a monomorphism.*

Here $C_r^*(\Gamma)$ is the reduced C^*-algebra of a group Γ, i.e. the completion of the group ring $\mathbb{C}\Gamma$ in the space of bounded linear operators on the Hilbert space $l_2(\Gamma)$ of complex square summable functions on Γ. Proofs of arrows from aNC can be found in [Ros],[FRR],[Con],[Ro2]. We note that the original version of aNC (due to Mischenko and Kasparov) was slightly weaker and it used the maximal C^*-algebra $C_m^*(\Gamma)$ of the group Γ.

Baum-Connes Conjecture (BCC). *The analytic assembly map* μ *is an isomorphism.*

Coarse Baum-Connes Conjecture (cBCC). *The coarse index map* $\mu : K_*^{lf}(X) \to K_*(C^*(X))$ *is an isomorphism, where* $X = E\Gamma$ *and* $C^*(X)$ *is the Roe algebra* [Ro2].

The connection of cBCC with BCC is based on the facts that the K-theory homology group $K_*(B\Gamma)$ is a Γ-equivariant K-homology of X and the reduced C^*-algebra of Γ is Morita equivalent to the algebra $C^*(X)^\Gamma$ of fixed elements of $C^*(X)$ under the action of Γ. The arrow cBCC $\to$ cNC is trivial. The arrow cBCC $\to$ aNC can be found in [Ro2].

Coarse Novikov Conjecture (cNC). *The coarse index map* μ *is a monomorphism.*

The arrow cNC $\to$ GLC is proven in [Ro1]. Here we consider a coarse analog of the analytic Novikov conjecture. For the L-theoretic coarse Novikov conjecture we refer to [DFW1] and [J].

Equivariant cNC. *The coarse index map* μ *is a* Γ*-equivariant split monomorphism.*

A proof of the arrow equi-cNC $\to$ NC is contained in [Ro2]. We give more attention to the following two conjectures.

Weinberger Conjecture (WC). *Let* $\bar{X} = X \cup \nu X$ *be the Higson compactification of* X*. Then the boundary homomorphism* $\delta : \check{H}^{n-1}(\nu X) \to H_c^n(X) = \mathbb{Z}$ *in the exact sequence of the pair* $(\bar{X}, \nu X)$ *is an epimorphism.*

We recall that for a smooth manifold X the *Higson compactification* $\overline{X}$ can be defined as the closure of the image of X under the diagonal embedding $\Phi : X \to I^{C_h(X)}$ into the Tychonov cube defined by means of all smooth functions $\phi : X \to I = [0,1]$ whose gradient tends to 0 as x goes to infinity. The set of all such ϕ is denoted by $C_h(X)$. the remainder $\nu X = \overline{X} \setminus X$ of the Higson compactification is called *the Higson corona*. The arrow WC $\to$ cNC was established in [Ro1]. The Weinberger Conjecture has the rational version (when coefficients are rational). The rational WC implies the Gromov Conjecture (actually after a stabilization when n is odd) [Ro1], [DF] and hence the Gromov-Lawson conjecture. There is an equivariant version of WC which states that δ is Γ-equivariant split epimorphism for cohomology L-theory. Weinberger noted that the rational equivariant WC implies NC [DF].

Gromov Conjecture (GC). *The manifold $X = E\Gamma$ is hypereuclidean.*

Gromov called this a 'problem' rather than a 'conjecture'. We use here GC instead of GP to make the picture more homogeneous. We recall that an n-dimensional manifold X is called *hypereuclidean* if it admits a proper 1-Lipschitz map $p : X \to \mathbb{R}^n$ of degree one. A manifold X is called *rationally hypereuclidean* if there exists a map p as above with $deg(p) \neq 0$. The arrow *rational* GC $\to$ GLC was proved in [GL], [G3]. The arrow GC $\to$ WC was proved by Roe [Ro1].

We note that the stable version of GC implies the Gromov-Lawson Conjecture as well. Also in [G3] there was an announcement of the implication (*stable*) GC $\to$ NC. Previously it was known that the equivariant version of GC implies the Novikov Conjecture [CGM]. The equivariant version of GC states that $X = E\Gamma$ is equivariantly hypereuclidean. the latter means that there is a equivariant map $p : X \times X \to R^n \times X$ which is 1-Lipschitz and essential on every fiber. The main example here is the universal cover of a closed manifold of nonpositive curvature. Then the map p is defined by the formula $p(x,y) = ln_y(x)$ where $ln_x : X \to T_x$ is the inverse of the exponential map at $x \in X$.

Example. All conjectures hold true when $\Gamma = \mathbb{Z}^n$. Then $B\Gamma$ is the n-dimensional torus and $X = \mathbb{R}^n$. Even in this toy case some of the above conjectures are not obvious.

We conclude the motivation part by a theorem of G. Yu [Yu1] (see also [Yu2], [HR2], [H], and [STY]).

Theorem 1.1. *If the asymptotic dimension* $\operatorname{asdim}\Gamma$ *of a finitely presented group* Γ *taken as a metric space with the word metric is finite, then the cBCC, and hence the NC, holds for* Γ.

This theorem was extended to cover the integral versions of the L- and K-theoretic Novikov conjectures in [CG], [CFY], [Ba], [DFW2].

1.2. **Coarse category and coarse structures.** The coarse category was defined by Roe in [Ro1]. He starts with the category whose objects are proper metric spaces. The morphisms are coarsely uniform, metric proper maps. Here are the definitions. A metric space X is called *proper* if every closed ball $B_r(x)$ in X is compact. We recall that a map $f : X \to Y$ is called *proper* if the preimage $f^{-1}(C)$ is compact for every compact set C. Then a metric space X is proper if and only if the distance to any fixed point is a proper function on X. A map $f : X \to Y$ is called *metric proper* if the preimage $f^{-1}(C)$ is bounded for every bounded set $C \subset Y$. A map $f : X \to Y$ is *coarsely uniform* if there is a tending to infinity function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that $d_Y(f(x), f(x')) \le \rho(d_X(x, x'))$ for all $x, x' \in X$. We consider the following equivalence relation on morphisms. Two maps $f, g : X \to Y$ are coarsely equivalent (bornotopic in terminology of [Ro1]) if there is a constant D such that $d_Y(f(x), g(x)) < D$ for all x. The coarse category is the quotient of the above category under this equivalence relation on the morphisms. Two metric spaces X and Y are coarsely equivalent if there are two morphisms $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is coarsely equivalent to 1_X and $f \circ g$ is coarsely equivalent to 1_Y.

Example. $\mathbb{Z}$ is coarsely equivalent to $\mathbb{R}$ with the metric $d(x, y) = |x - y|$.

More generally, if $B\Gamma$ is a finite complex, then Γ is coarsely equivalent to $E\Gamma$. Here the metric on $E\Gamma$ is lifted from one on $B\Gamma$ and the group Γ is equipped with the word metric with respect to a finite set of generators. We recall that if $S = S^{-1}$ is a finite symmetric set of generators of a group Γ then the *word metric* d_S is defined as $d_S(x, y) = \|x^{-1}y\|_S$, where the S-norm $\|a\|_S$ of an element $a \in \Gamma$ is the shortest length of presentation of a in the alphabet S. We note that if S' is another finite symmetric generating set of Γ, then the metric spaces (Γ, d_S) and $(\Gamma, d_{S'})$ are coarsely equivalent.

We call a metric space X *ϵ-discrete* if $d_X(x, x') \ge \epsilon$ for all $x, x' \in X$, $x \ne x'$. We call it *discrete* if it is ϵ-discrete for some ϵ.

Proposition 1.2. *Every metric space X is coarsely equivalent to a discrete metric space.*

Proof. By transfinite induction one can construct a 1-discrete subset $S \subset X$ with the property $d_X(x, S) \le 1$ for all $x \in X$. The inclusion $S \subset X$ is a coarse equivalence whose inverse is any map $g : X \to S$ with the property $d(x, g(x)) \le d(x, S) + 1$. □

We are going to study a coarse invariant dimension on metric spaces. Before giving the definitions we will sketch an approach to an extension of the coarse category beyond the metric spaces which is due to Higson and Roe [HR].

A set X is given a *coarse structure* if for every set S there is a fixed equivalence relation on the set of maps X^S called being *close* and satisfying the following axioms:

(1) If $p_1, p_2 : S \to X$ are close, then $p_1 \circ q$ and $p_2 \circ q$ are close for every $q : S' \to S$;
(2) If $p_1, p_2 : S \to X$ are close and $q_1, q_2 : S' \to X$ are close, then $p_1 \coprod q_1$ and $p_2 \coprod q_2$ are close maps of $S \coprod S'$ to X;
(3) any two constant maps are close.

A subset $C \subset X$ is called bounded (with respect to the coarse structure on X) if the inclusion map $i : C \to X$ is close to a constant map. A map $f : X \to Y$ between two coarse spaces is called *coarse proper* if the preimage of every bounded set is bounded. Then morphisms between coarse spaces are coarse proper maps $f : X \to Y$ satisfying the condition:

$$p_1, p_2 : S \to X \ \text{ are close } \ \Rightarrow \ f \circ p_1, f \circ p_2 : S \to Y \ \text{ are close.}$$

Examples. (1) When X is a metric space one sets for being close the property to be in a finite distance.

(2) If a locally compact topological space X is embedded in its compactification $\bar{X}$, one can define two maps $p_1, p_2 : S \to X$ to be close if for every subset $S' \subset S$ the corresponding limit sets coincide: $\overline{p_1(S')} \setminus X = \overline{p_2(S')} \setminus X$.

We denote by $\mathbb{R}^n_r$ the coarse structure on $\mathbb{R}^n$ defined by the radial compactification.

In parallel with the bounded and continuous control in controlled topology [FP], the coarse structure on X defined in (1) is called *bounded* and the coarse structure defined in (2) is called *continuous*.

Proposition 1.3. *The bounded coarse structure on a proper metric space X coincides with the continuous coarse structure generated by the Higson compactification.*

The proof can be easily derived from the following description of the Higson compactification. According to Smirnov's theorem every compactification on X is defined by some proximity (and vice versa). The Higson corona of X is defined by the proximity δ_X given by the condition $A \bar{\delta}_X B$ if and only if $\lim_{r \to \infty} d_X(A \setminus B_r(x_0), B \setminus B_r(x_0)) = \infty$. It means that the closures of diverging sets in X (and only them) do not intersect in the Higson corona.

Lecture 2. THEOREMS

2.1. **Definitions.** There are several equivalent definitions of dimension of compact metric spaces. The equivalence of corresponding coarse analogs for proper metric spaces in some cases is still an open question.

We recall the terminology. Let $\mathcal{U}$ denote an open cover of a metric space X. Then $ord(\mathcal{U})$ is the *order* of the cover, i.e. the maximal number of elements of $\mathcal{U}$ having nonempty intersection. The *mesh* of a cover $\mathcal{U}$, $mesh(\mathcal{U})$, is the maximal diameter of the elements of $\mathcal{U}$. The *Lebesgue number* of a cover $\mathcal{U}$ is defined as $L(\mathcal{U}) = \inf_{y\in Y}\sup_{U\in\mathcal{U}} d(y, Y \setminus U)$. A family $\mathcal{U}$ of subsets of X is called *uniformly bounded* if there is an upper bound on the diameter of its elements.

We consider the following comparison table:

Dimension $\dim X \le n$	**Asymptotic dimension** $\operatorname{asdim} X \le n$
(1) $\forall\, \mathcal{V}$, open cover of X, $\exists\, \mathcal{U}$, an open cover of X, with $ord(\mathcal{U}) \le n+1$ and $\mathcal{U} \prec \mathcal{V}$.	(1) $\forall\, \mathcal{V}$, uniformly bounded cover of X, $\exists\, \mathcal{U}$, a uniformly bounded cover of X, with $ord(\mathcal{U}) \le n+1$ and $\mathcal{V} \prec \mathcal{U}$.
(2) $\forall\ \epsilon > 0\ \exists\ \mathcal{U}^0, \dots, \mathcal{U}^n$, disjoint families of sets in X with $mesh(\mathcal{U}^i) < \epsilon$ such that $\cup_i \mathcal{U}^i$ is a cover of X.	(2) $\forall\ \lambda\ \exists$ uniformly bounded λ-disjoint families $\mathcal{U}^0, \dots, \mathcal{U}^n$ such that $\cup_i \mathcal{U}^i$ is a cover of X.
(3) $\forall\ \epsilon > 0\ \exists$ an ϵ-map $f : X \to K$ to an n-dimensional polyhedron K.	(3) $\forall\ \lambda\ \exists$ uniformly cobounded 1-Lipschitz map $f : X \to K$ to a uniform polyhedron K with $\dim K = n$ and $mesh(K) = \lambda$.
(4) X admits a Čech approximation by n-dimensional polyhedra.	(4) X admits an anti-Čech approximation by n-dimensional polyhedra.
(5) $\forall\ f : A \to S^n$, $A \subset_{Cl} X$, $\exists$ an extension $\bar{f} : X \to S^n$.	(5) $\forall\ f : A \to \mathbb{R}^{n+1}_r$, $A \subset_{Cl} X$, $\exists$ an extension $\bar{f} : X \to \mathbb{R}^{n+1}_r$.
(6) $\operatorname{Ind} X \le n$	(6) $\operatorname{asInd} X \le n$.

In the column on the left we have equivalent definitions of dimension for compact metric spaces. In the right column there are asymptotic counterparts. It is likely that they all are equivalent for metric spaces with bounded geometry.

We still owe some definitions for the asymptotic part of this table. A map $f : X \to Y$ between metric spaces is called *uniformly cobounded* if for every $R > 0$ the diameter of the preimage $f^{-1}(B_R(y))$ is uniformly bounded from above. A *Čech approximation* of a compact metric space X is a sequence of finite covers $\{\mathcal{U}_n\}$ such that, $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{U}_n$ for all n, and $\lim_{n\to\infty} mesh(\mathcal{U}_n) = 0$. An anti-Čech approximation [Ro1] of a metric space X is a sequence of uniformly bounded locally finite covers $\mathcal{U}_n$ such that $\mathcal{U}_n$ is a refinement of $\mathcal{U}_{n+1}$, and $\lim_{n\to\infty} L(\mathcal{U}_n) = \infty$. In both cases the approximation of metric space X is given by polyhedra which are nerves of corresponding covers. We say that a simplicial complex K is given a *uniform metric* of $mesh(K) = \lambda$, if it is realized as a subcomplex in the standard λ-simplex Δ_λ in the Hilbert space l_2

$$\Delta_\lambda = \{(x_i) \mid \sum x_i = \lambda,\ x_i \geq 0\}$$

and it's metric is induced from l_2.

In condition (5), $\mathbb{R}^{n+1}_r$ stands for the continuous coarse structure on $\mathbb{R}^{n+1}$ defined by the radial compactification. Since every coarse morphism $f : A \to \mathbb{R}^{n+1}_r$ defines a continuous map between coronas $f : \nu A \to S^n$ and vice versa, the asymptotic condition (5) (in view of the classical condition (5)) can be reformulated as follows:

(5′) $\dim \nu X \leq n$.

We note that the dimension dim of a nonmetrizible compact space can be defined by the condition (5).

Finally we recall the definition of inductive dimensions. A closed subset C of a topological space X is called a *separator* between disjoint subsets $A, B \subset X$ if $X \setminus C = U \cup V$, where U, V are open subsets in X, $U \cap V = \emptyset$, $A \subset U$, $V \subset B$. We set $\operatorname{Ind}\emptyset = -1$. Then $\operatorname{Ind} X \leq n$ if for every two disjoint closed sets $A, B \subset X$ there is a separator C with $\operatorname{Ind} C \leq n - 1$ [En].

It is known that the Higson corona is a functor from the category of proper metric spaces and coarse maps into the category of compact Hausdorff spaces and continuous maps. In particular, if $X \subset Y$, then $\nu X \subset \nu Y$. For any subset A of X we denote by A' its trace on νX, i. e. the intersection of the closure of A in $\bar{X}$ with νX. Obviously, the set A' coincides with the Higson corona νA. Let X be a proper metric space. Two sets A, B in a metric space are called *asymptotically disjoint* if the traces A', B' on νX are disjoint. A subset C of a metric space X is an *asymptotic separator* between asymptotically disjoint subsets $A, B \subset X$ if the trace C' is a separator in νX between A' and B'. By the definition, $\operatorname{asInd} X = -1$ if and only if X is bounded. Suppose we have defined the class of all proper metric spaces Y with $\mathrm{asInd}\, Y \le n-1$. Then $\mathrm{asInd}\, X \le n$ if and only if for every asymptotically disjoint subsets $A, B \subset X$ there exists an asymptotic separator C between A and B with $\mathrm{Ind}\, C \le n-1$. The dimension functions asInd is called the *asymptotic inductive dimension* .

As it was mentioned, all conditions (1)–(6) in the left column are equivalent for compact metric spaces [HW],[En]. The condition (1) is Lebesgue's definition of dimension. The equivalence (1) $\Leftrightarrow$ (2) is a theorem of Ostrand. The equivalence of the conditions (3) and (5) to the inequality $\dim X \le n$ is due to Alexandroff. The equivalence $\dim X \le n \Leftrightarrow$ (4) is called the Froudenthal theorem.

In the column on the right Gromov proved the equivalence of conditions (1),(2),(3) and (4) [Gr1] (see [BD2] for details). These conditions give a definition of the asymptotic dimension asdim. In [Dr1] it was shown that the condition (5′) is equivalent to the inequality $\mathrm{asdim}\, X \le n$ provided $\mathrm{asdim}\, X < \infty$. Under the same condition the equality $\mathrm{asInd}\, X = \mathrm{asdim}\, X$ was proven in [Dr7], [DZ]. Here we exclude the case of bounded X. We note that there are implications (1) $\Rightarrow$ (5′) [DKU], (1) $\Rightarrow$ (6) [DZ]. The status of the remaining implications is unknown.

Examples.
(1) $\mathrm{asdim}\, \mathbb{Z} = 1$;
(2) $\mathrm{asdim}\, \mathbb{R}^n = n$ [DKU];
(3) $\mathrm{asdim}\, T = 1$ where T is a tree (with the natural metric).

We note that all asymptotic conditions (1)–(6) are coarse invariant. All of them can be stated in the setting of general coarse structures. To do that one needs a notion of a uniformly bounded family of sets in a general coarse space. A family $\mathcal{U}$ in a coarse space X is *uniformly bounded* if the maps $p_1, p_2 : S \to X$ are close, where $S = \cup_{U \in \mathcal{U}} U \times U \subset X \times X$ and $p_1, p_2 : X \times X \to X$ are the projections onto the first and the second factors respectively.

2.2. **Embedding Theorems and Applications.** A coarse morphism $f : X \to Y$ is a *coarse embedding* if there the inverse morphism defined for $f : X \to f(X)$, i.e. a morphism $g : f(X) \to X$ such that $g \circ f$ and 1_X are close in X and $f \circ g$ and $1_{f(X)}$ are close in $f(X)$ with respect to the induced coarse structure. If a coarse morphism f is injective in the set theoretic sense, then it is a coarse embedding if and only f^{-1} is a coarse morphism. In our metric setting a map $f : X \to Y$ is a coarse embedding if there are tending to infinity functions $\rho_1, \rho_2 : \mathbb{R}_+ \to \mathbb{R}_+$

such that

$$\rho_1(d_X(x,x')) \leq d_Y(f(x), f(x')) \leq \rho_2(d_X(x,x')).$$

We recall that a metric space (X, d_X) is called *geodesic* if for every pair of its points x and y there is an isometric embedding of the interval $[0, d(x,y)]$ into X with the end points x and y. Clearly for a geodesic metric space X the function ρ_2 can be taken linear. Thus, up to a rescaling, a coarse embedding of a geodesic metric space is an 1-Lipschitz map.

The question about embeddings into nicer spaces in the coarse category is very important for applications. In [Yu2] Goulang Yu proved the Novikov Conjecture for groups Γ that admit a coarse imbedding in the Hilbert space (see aloso [H] and [STY]). It was noticed in [HR2] that a metric space with finite asymptotic dimension is coarsely imbeddable in l_2. Thus this theorem of Yu implies his Theorem 1.1.

In a geometric approach to Theorem 1.1 the need for a coarse analog of the classical Nobeling-Pontryagin embedding theorem arose. We recall that the classical Nobeling-Pontryagin embedding theorem states that every compactum X of dimension $\dim X \leq n$ can be embedded in $\mathbb{R}^{2n+1}$. It is easy to see that this statement does not have a direct asymptotic analog. Indeed, a binary tree being asymptotically 1-dimensional cannot be coarsely embedded in R^N for any N because the tree has an exponential volume growth function and a euclidean space has only the polynomial volume growth. Moreover, we show in [DZ] that there is no metric space of bounded geometry that contains in a coarse sense all asymptotically n-dimensional metric spaces of bounded geometry. Here the bounded geometry condition serves as an asymptotic analog of compactness.

We recall that the *ϵ-capacity* $c_\epsilon(W)$ of a subset $W \subset X$ of a metric space X is the maximal cardinality of ϵ-discrete set in W. A metric space X has *bounded geometry* if there are $\epsilon > 0$ and a function $c : \mathbb{R}_+ \to \mathbb{R}_+$ such that $c_\epsilon(B_r(x)) \leq c(r)$ for all $x \in X$. Finitely generated groups give us one of the main sources of examples of metric spaces of bounded geometry.

Neveretheless in asymptotic topology there is an embedding theorem which turnes out to be sufficient for the purpose of Theorem 1.1.

Theorem 2.1 ([Dr4]). *Every metric space of bounded geometry X with* $\operatorname{asdim} X \leq n$ *can be coarsely embedded in a $2n+2$-dimensional manifold of nonpositive curvature.*

The proof is based on the following embedding theorem.

Theorem 2.2 ([Dr4]). *Every metric space of bounded geometry X with* $\operatorname{asdim} X \leq n$ *can be coarsely embedded in the product of $n+1$ locally finite trees.*

In the classical dimension theory there is a theorem [Bow] analogous to Theorem 2.2 which states that an n-dimensional compact metric space can be imbedded in the product of $n+1$ dendrits (= 1-dimensional AR).

We recall that in the classical dimension theory for every n there is the universal Menger compactum μ^n which is n-dimensional and contains a copy of every n-dimensional compactum. As we mentioned, there is no similar object in the coarse category for $n > 0$ [DZ]. Using an embedding $X \to \prod T_i$ into the product of trees as in Theorem 2.2 we built a coarse analog of the Menger space $M(\{T_i\})$, $\operatorname{asdim} M(\{T_i\}) = n$, out of this product and get an embedding of X into $M(\{T_i\})$. This construction leads to the universal space for asdim $\leq n$ but we lose the bounded geometry condition.

For $n = 0$ a universal object with bounded geometry does exist. It is a literal generalization of the Cantor set: M^0 is the subset of all reals that do not use 2 in their ternary expansion. The classical Cantor set is $M^0 \cap [0,1]$.

We proved a stable version of the Gromov Conjecture (see §1) for a group Γ with $\operatorname{asdim} \Gamma < \infty$.

Theorem 2.3 ([Dr2]). *Let M be a closed aspherical manifold with* $\operatorname{asdim} \pi_1(M) < \infty$ *and let X be its universal cover. Then the manifold $X \times \mathbb{R}^m$ is hypereuclidean for some m.*

A weaker theorem states that $X \times \mathbb{R}^m$ is integrally hyperspherical [Dr4]. This theorem enables us to prove the GLC. There is a relatively short proof of this which is based on the Theorem 2.1. We recall that an n-dimensional manifold Y is integrally hyperspherical [GL] if for arbitrary large r there is an n-submanifold with boundary $V_r \subset Y$ and an 1-Lipschitz degree one map $p_r : (V_r, \partial V_r) \to (B_r(0), \partial B_r(0))$ to the euclidean ball of radius r. If X is embedded in a k-dimensional nonpositively curved manifold W^k, the R-sphere $S_R(x_0)$ in X for large enough R is linked with a manifold M which has a sufficiently large tubular neighborhood N in W^k also linked with $S_R(x_0)$ and with an 1-Lipschitz trivialization $\pi : N \to B_r(0)$. Then we take a general position intersection $X \cap N$ as V_r and the restriction $\pi|_{V_r}$ as p_r. Crossing with $\mathbb{R}^m$ helps to achieve the above properties of the tubular neighborhood N.

When Gromov defined the asymptotic dimension [G1] he already suggested to consider the asymptotic behavior of some natural functions that appeared in the definition as legitimate asymptotic invariants of dimension type. Here we consider one of such functions defined as

$$asd_X(\lambda) = \min\{ord(\mathcal{U})|\ L(\mathcal{U}) \geq \lambda\} - 1,$$

where $\mathcal{U}$ is a uniformly bounded cover of X. We note that taking the limit gives the equality:

$$\lim_{\lambda\to\infty} asd_X(\lambda) = \operatorname{asdim} X.$$

So we will refer to the function $asd_X(\lambda)$ as to the asymptotic dimension of X in the case when $\operatorname{asdim} X = \infty$. Clearly, for a space of bounded geometry the function $asd_X(\lambda)$ is at most exponential. The following is a generalization of the theorem of Yu (Theorem 1.1).

Theorem 2.4 ([Dr5], [Dr8])**.** *If $asd_\Gamma(\lambda)$ has the polynomial growth, then the Novikov Conjecture holds for Γ.*

This theorem holds for all finitely presented groups Γ. In contrast with Theorem 2.3, the proof here relies heavily on the results of [Yu2], [STY], and [H].

2.3. **Finite dimensionality theorems.** Finite dimensionality results for groups are important for the application to the Novikov Conjecture. The first finite dimensionality result in the asymptotic dimension theory is due to Gromov who proved that $\operatorname{asdim}\Gamma < \infty$ for hyperbolic groups [Gr1], [Ro3]. Then we proved in [DJ] that $\operatorname{asdim}\Gamma < \infty$ for all Coxeter groups. In [BD1] we proved that the asymptotic finite dimensionality is preserved by the amalgamated product and by the HNN extension. We gave a general estimate.

Theorem 2.5 ([BD2])**.** *Suppose that Γ is the fundamental group of a finite graph of groups with all vertex groups G_v having $\operatorname{asdim} G_v \leq n$. Then $\operatorname{asdim}\Gamma \leq n+1$.*

A *graph of groups* is a graph in which every vertex v and every edge e have assigned group G_v and G_e such that for the endpoints $e^\pm$ of e there are fixed monomorphisms $\phi_{e^\pm} : G_e \to G_{e^\pm}$. The fundamental group of a graph of groups can be viewed as the fundamental group of a complex built out of the mapping cylinders of the maps between Eilenberg-Maclane complexes $f_{e^\pm} : K(G_e, 1) \to K(G_{e^\pm}, 1)$ defined by the homomorphisms $\phi_{e^\pm}$. Clearly, this is a generalization of the amalgamated product and the HNN extension which correspond to the graphs with one edge.

By Bass-Serre theory the fundamental groups of graphs of groups are exactly the groups acting on trees (without inversion). We used this action to obtain our estimate. We proved the following theorem.

Theorem 2.6 ([BD2]). *Suppose that a group Γ acts by isometries on a metric space X with* $\operatorname{asdim} X \leq k$ *in such a way that for every r, the r-stabilizer $W_r(x_0)$ of a fixed point $x_0 \in X$ has* $\operatorname{asdim} W_r(x_0) \leq n$. *Then* $\operatorname{asdim} \Gamma \leq n + k$.

We define the *r-stabilizer* $W_r(x_0)$ as the set

$$\{g \in \Gamma |\ d_X(g(x_0), x_0) \leq r\}.$$

Thus, to prove Theorem 2.5 it suffices to show that $\operatorname{asdim} W_r(x_0) \leq n$ for the Serre action of the group Γ on a tree. It is not an easy task by any means even in the simplest case of the free product of groups. the difficulties were overcome by further development of the asymptotic dimension theory. We proved the following union theorem.

Theorem 2.7 ([BD1]).

(1) *Suppose $X = A \cup B$ is a metric space. Then* $\operatorname{asdim} X \leq \max\{\operatorname{asdim} A, \operatorname{asdim} B\}$;
(2) *Suppose $X = \cup_i A_i$ is a metric space and let* $\operatorname{asdim} A_i \leq n$ *for all i. Then* $\operatorname{asdim} X \leq n$ *provided the following condition is satisfied: $\forall r\ \exists\ Y_r \subset X$ with* $\operatorname{asdim} Y_r \leq n$ *such that the family of sets $\{A_i \setminus Y_r\}$ is r-disjoint.*

We note that these union theorems differ from their classical analogs.

Using the asymptotic inductive dimension asInd we managed to get an exact formula in the case of the nondegenerate amalgamated product.

Theorem 2.8 ([BDK]). *There is a formula*

$$asdim A * B = \max\{asdim A, asdim B, 1\}$$

for finitely generated groups A and B.

For the amalgamated product, the best what we have is the inequality [BD3]

$$asdim A *_C B \leq \max\{asdim A/C, asdim B/C, asdim C + 1\}.$$

Lecture 3. COUNTEREXAMPLES

3.1. **Coarse Alexandroff Problem.** We recall that the classical Alexandroff problem was about coincidence of the integral cohomological dimension of a compact metric space with its dimension. Since the 1930s the problem was reduced to the question whether there is an infinite dimensional compactum with a finite cohomological dimension. The problem was solved negatively [Dr6]. We recall that the cohomological dimension of X is defined in terms of Čech cohomology as the maximal number n such that the relative cohomology group is non-trivial, $\check{H}(X, A; \mathbb{Z}) \neq 0$, for some closed subset $A \subset X$. The Čech cohomology is defined by means of a Čech approximation of a compactum X (or of a pair (X, A)) and the ordinary (simplicial) cohomology. Similarly one can define the anti-Čech homology called *coarse homology* of a metric space (or pair) [Ro2], [Dr1] by means of an anti-vCech approximation of a metric space X (or of a pair (X, A)) and the simplicial homology with infinite chains. Roe denoted the coarse homology as HX_* [Ro1]. Then using the coarse homology one can define an asymptotic homological dimension in a similar fashion: $\operatorname{asdim}_{\mathbb{Z}} X = \max\{n | HX_n(X, A) \neq 0, A \subset_{Cl} X\}$. The homology is more preferable here than the cohomology since the latter involves the $\lim^1$ term. By analogy we can pose the coarse version of Alexandroff problem:

Coarse Alexandroff Problem. *Does there exist an asymptotically infinite dimensional metric space with a finite asymptotic homological dimension?*

In view of Yu's theorem (Theorem 1.1), it is not difficult to show (see [Dr1]) that the negative answer to this problem implies the Novikov Conjecture for the groups Γ with $B\Gamma$ a finite complex.

The following was the first counterexample to the coarse Alexandroff Problem, though it appeared as a counterexample to a general version of the Gromov Conjecture (GC) as well as to a preliminary version of the coarse Baum-Connes conjecture [Ro1].

Counterexample 3.1 ([DFW1])**.** *There exists a uniformly contractible Riemannian metric on* $\mathbb{R}^8$ *which gives a metric space* X *with* $\operatorname{asdim} X = \infty$ *and with the asymptotic homological dimension equal to 8.*

In our paper we proved that X is not stably hypereuclidean. This already implies that $\operatorname{asdim} X = \infty$. Higson and Roe proved [HR1] that for uniformly contractible spaces the coarse homology coincides with the locally finite homology. This gives us the required estimate for asymptotic homological dimension.

We recall that a metric space X is *uniformly contractible* if there is a function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that every r-ball $B_r(x)$ in X can be contracted to a point in $B_{\rho(r)}(x)$. We note that the universal cover of a closed aspherical manifold is always uniformly contractible. This let Gromov to pose his conjecture GC for all uniformly contractible manifolds [G2]. The counterexample 3.1 disproves GC in the general setting but not the rational GC. The construction of it is based on a (dimension raising) cell-like map of a 7-dimensional sphere which has non-zero kernel in the homology K-theory. Since rationally a cell-like map is always an isomorphism, our approach did not touch the rational GC.

The drawback of this counterexample is that X is not a $E\Gamma$ and moreover X does not have bounded geometry.

Recently Gromov came with a better example.

Counterexample 3.2 ([G5]). *There is a closed aspherical manifold M with* $\operatorname{asdim} \pi_1(M) = \infty$.

We note that the universal cover X of M, as well as the fundamental group $\pi_1(M)$, has the asymptotic homological dimension $\operatorname{asdim}_{\mathbb{Z}} X$ equal to the dimension of M (=4 in the most recent version). Gromov's construction is based on use of expander. He constructed his manifold M with $\pi_1(M)$ containing an expander in a coarse sense. Then the equality $\operatorname{asdim} \pi_1(M) = \infty$ follow (see the next section).

3.2. **Expanders.** Let (V, E) be a finite graph with the vertex set V and the edge set E. We denote the cardinalities $|V|$ and $\|E\|$ by n and m. Let $l_2(V)$ and $l_2(E)$ denote complex vector spaces generated by V and E. We view an element of $l_2(V)$ as a function $f : V \to \mathbb{C}$. We fix an orientation on E and define the differential $d : l_2(V) \to l_2(E)$ as $(df)(e) = f(e^+) - f(e^-)$. The operator d is represented by $m \times n$ matrix D. We define the Laplace operator $\Delta = D^*D$ where D^* is the transpose of D. It is an easy exercise to show that Δ does not depend on orientation on E. By the definition the operator Δ is self-adjoint. Also it is positive: $\langle \Delta f, f\rangle = \langle Df, Df\rangle \geq 0$. Therefore Δ has real nonnegative eigenvalues. We denote by $\lambda_1(V)$ the minimal positive eigenvalue of the laplacian on the graph V.

Definition. A sequence of graphs (V_n, E_n) of a fixed valency d and with $|V_n| \to \infty$ is called an **expander** (or expanding sequence of graphs) if there a positive constant c such that $\lambda_1(V_n) \geq c$ for all n.

The last condition on the graphs is equivalent to the following [Lu]: *there is a constant* $c_0 > 0$ *such that* $|\partial A| \geq c_0|A|$ *for all subsets* $A \subset V_n$ *with* $|A| \leq |V_n|/2$.

Here the boundary of A in a graph V is defined as

$$\partial A = \{x \in V \mid dist(x, A) = 1\}.$$

It is easy to prove that the solutions of the Laplace equation $\Delta f = 0$ are exactly constant functions. The orthogonal space to the constants we denote by $l_2^0(V) = \{f \mid \sum_{v\in V} f(v) = 0\}$. We consider the restriction

Δ to $l_2^0(V)$. Let $\{v_i\}$ be a orthonormal basis of eigenvectors in $l_2^0(V)$ and let $f = \sum \alpha_i v_i$. Then

$$\frac{\langle \Delta f, f\rangle}{\langle f, f\rangle} = \frac{\langle \sum \lambda_i \alpha_i v_i, \sum \alpha_i v_i\rangle}{\langle \sum \alpha_i v_i, \sum \alpha_i v_i\rangle} = \frac{\sum \lambda_i \alpha_i^2}{\sum \alpha_i^2} \geq \frac{\sum \lambda_1 \alpha_i^2}{\sum \alpha_i^2} = \lambda_1.$$

We apply the above inequality to a real-valued function $f : V \to \mathbb{R}$, $f \in l_2^0(V)$ to obtain the following inequality:

$$\lambda_1 \leq \frac{\langle \Delta f, f\rangle}{\langle f, f\rangle} = \frac{\langle df, df\rangle}{\langle f, f\rangle} = \frac{\sum_E |f(e^+) - f(e^-)|^2}{\sum_V |f(x)|^2}.$$

We rearrange this inequality into the inequality $\lambda_1 \sum_V |f(x)|^2 \leq \sum_E |f(e^+) - f(e^-)|^2$. Clearly this inequality holds for any function $f : V \to l_2$ to a Hilbert space such that $\sum_V f(v) = 0$:

$$\lambda_1 \sum_V \|f(x)\|^2 \leq \sum_E \|f(x) - f(y)\|^2.$$

Since $m = |E| = dn/2$, we can change the above inequality into the following

$$\lambda_1 \frac{1}{|V|} \sum_V \|f(x)\|^2 \leq \frac{d}{2|E|} \sum_E \|f(x) - f(y)\|^2.$$

On the right we have $d/2$ times average of squares of lengths of the images under f of edges in the graph. Applying this inequality to an 1-Lipschitz map and using the estimate $\lambda_{\geq} c$ we obtain the following.

Proposition 3.3. *Let $f_n : V_n \to l_2$ be a sequence of 1-Lipschitz maps of an expander to a Hilbert space. Then*

$$\frac{1}{|V|} \sum_V \|f_n(x)\|^2 \leq \frac{d}{2c}$$

for all n.

Corollary 3.4. *If $K > \sqrt{d/c}$, then for maps $f_n : V_n \to l_2$ as above there is the inequality $|\{x \in V_n \mid \|f_n(x)\| \leq K\}| > |V_n|/2$ for all n.*

Proof. Assume the contrary. Then we have a contradiction

$$\frac{d}{2c} \geq \frac{1}{|V|} \sum_V \|f_n(x)\|^2 \geq \frac{1}{|V|} K^2 \frac{|V|}{2} > \frac{d}{2c}.$$

$\square$

Nice groups cannot contain (in the coarse sense) an expander. We proved the following

Theorem 3.5 ([Dr3]). *Suppose that the universal cover X of a closed aspherical manifold is equivarinatly hypereuclidean, then X does not contain an expander.*

Corollary 3.6. *Gromov's example (Counterexample 3.2) is a counterexample to the equivariant Gromov Conjecture (equi-GC) and to the equivariant Weinberger Conjecture (equi-WC).*

Here we give a proof of a weaker statement which is due to Gromov and Higson. Namely we show that

A contractible Riemannian manifold with a nonpositive sectional curvature does not contain an expander.

We note that in view of this result Theorem 2.1 implies that every space containing an expander has infinite asymptotic dimension.

We present Higson's argument here.

Proof. Let $\dim X = m$ and let $\{V_n\}$ be an expander that lies in X. By Hadamard theorem the exponent $\exp_x : T_x \to X$ is a diffeomorphism for every $x \in X$. We note that the inverse map $\log_x : X \to T_x = \mathbb{R}^m$ is 1-Lipschitz.

First we show that for every n there is a point y_n such that

$$\sum_{x \in V_n} \log_{y_n}(x) = 0.$$

Assume the contrary $w_y = \sum_{x \in V_n} \log_y(x) \neq 0$ for all $y \in X$. Then the vector $-w_y$ defines a point $s_y \in S(\infty)$ in the visual sphere at infinity $S(\infty)$ of a manifold X. It is not difficult to check that the correspondence $y \to s_y$ defines a continuous map $f : X \to S(\infty)$ which is a retraction of the topological m-ball $X \cup S(\infty)$ to its boundary. This is a contradiction.

We take K as in Corollary 3.4. Then

$$|(\log_{y_n})^{-1}(B_K(0)) \cap V_n| > \frac{|V_n|}{2}.$$

Since $(\log_{y_n})^{-1}(B_K(0)) = \exp_{y_n}(B_K(0)) = B^X_K(y_n)$, where the latter is the K-ball in X, we have an estimate

$$2d^{2K} \geq 1+d+\cdots+d^{2K} \geq |B^{V_n}_{2K}(v)| = |B^X_{2K}(v) \cap V_n| \geq |B^X_K(y_n) \cap V_n| > \frac{|V_n|}{2}$$

for any $v \in B^X_K(y_n) \cap V_n$. This gives a contradiction with $|V_n| \to \infty$. $\square$

In conclusion we note that the Novikov Conjecture holds true for this Gromov's group.

Another remark is that the Higson corona of an expander, considered metrically as a garland of finite graphs attached to a half-line, might produce by means of factorization dimensionally exotic metric compacta. It could give a clue to some long standing problems in infinite dimensional dimension theory.

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