ON CERTAIN I-D COMPACTA

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ABSTRACT. Three examples of nontypical i-d compacta are presented. An application to absorbers follows.

1. NONTYPICAL COMPACTA

The typical property, for an i-d compactum K, is that K is homeomorphic to its square, that is,

$$K \cong K \times K.$$

Here are the properties that are stronger than the *negation* of the above:

(1) No open subset of $K \times K$ can be embedded into $K \times I^q$ for any q (I stands for $[-1, 1]$).
(2) $K \times K$ cannot be embedded into $K \times \sigma$; $\sigma = \bigcup_{q=1}^{\infty} I^q \subset Q = I^{\infty}$.
(3) $K \times K$ cannot be embedded into $K \times I^q$ for any q.
(4) $K \times K$ cannot be embedded into K.

Definition. A map $K \times K \supset A \to Z$ is *fiberwise injective (f-i)* if restricted to every fiber $\{k\} \times K$ or $K \times \{k\}$ it is injective.

Fact 1. If K is carries either a group structure or a convex structure then $K \times K$ admits a f-i map into K. The maps

$$(x, y) \to xy$$

or

$$(x, y) \to \frac{1}{2}(x + y)$$

are easily seen to be f-i.

Here are counterparts of properties (1)-(4):

(1′) No open set U of $K \times K$ admits a f-i map into $Z = K \times I^q$ for any q.
(2′) There is no f-i map $K \times K \to Z = K \times \sigma$.
(3′) There is no f-i map $K \times K \to Z = K \times I^q$ for any q.
(4′) There is no f-i map $K \times K \to K$.

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For a compactum K, we have the following implications

$$1' \Rightarrow 1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$$

and

$$1' \Rightarrow 2' \Rightarrow 3' \rightarrow 4'$$

The implications $1 \Rightarrow 2$ and $1' \Rightarrow 2'$ follow from the Baire category theorem applied to $K \times K$ (having in mind that $K \times \sigma = \bigcup_{q=1}^{\infty} K \times I^q$).

Furthermore, we have

Remark 1. Assume $K \subset Z$ and Z is a countable union of compacta embeddable in $K \times \sigma$. If K satisfies property (1) (resp., property (1′)), then Z satisfies (2) (resp., (2′)); consequently, $Z \times Z$ is not embeddable in Z (resp., there is no f-i map $Z \times Z \rightarrow Z$).

In what follows we will discuss examples that were presented in [D] (see also [BC]).

Example 1. Let C be Cook's continuum, that is, C is hereditarily indecomposable continuum and, for every continuum $A \subset C$, every map $A \rightarrow C$ is either constant or an inclusion. Every compactum of the form

$$P = \prod_{i=1}^{\infty} A_i,$$

where $A_i \subset C$ are pairwise disjoint subcontinua, satisfies property (1). Moreover, P (and every open subset of P) is strongly infinite-dimensional and contains subsets of all finite dimensions.

Example 2. Let us recall that the Smirnov Cubes S_α, $\alpha < \omega_1$, are compacta defined as follows $S_0 = \{0\}$, $S_{\beta+1} = S_\beta \times I$; and, for a limit ordinal α, $S_\alpha = \omega(\oplus_{\beta<\alpha} S_\beta)$, the one-point compactification of S_β. For, for $\alpha_0 = \omega^\omega$, the space

$$S = S_{\alpha_0}$$

satisfies (3).

Proof. This follows from the fact that $\mathrm{trind}(S_{\alpha_0} \times S_{\alpha_0}) = \alpha_0(+)\alpha_0$ and $\mathrm{trind}(S_\alpha \times I^q) \leq \alpha(+)q$, where trind stands for the small transfinite inductive dimension. □

The next example is due to J. Kulesza.

Example 3. The space

$$T = \omega((\oplus_{n \geq 1} I^n) \oplus H),$$

where H is a hereditary i-d continuum, has property (3′).

Proof. Let $f : T \times T \to T \times I^q$ be f-i. Then $f(H \times I^k) \subset H \times I^q$ for $k > q$. In particular, I^k embeds into $H \times I^q$. Since $k > q$ and the projection is closed, there exists a fiber $H \times \{x\} \subset H \times I^q$ containing a closed set with dim > 0, a contradiction. □

Congesting singularities. Write L for either S or T. Pick a null-sequence $\{C_n\}$ of pairwise disjoint Cantor sets in the Cantor set C so that every open nonempty subset of C contains some C_n. Let $f_n : C_n \to L$, be a surjection. Define $\tilde{S}$ (resp., $\tilde{T}$) to be the adjoint space with S (resp., T) attached in place of each C_n via the map f_n.

Fact 2. The compactum $\tilde{S}$ satisfies property (1); moreover, it is countable-dimensional and $\text{trind}(\tilde{S}) \leq \text{trind}(S) + 1$. The compactum $\tilde{T}$ is **not** countable dimensional and satisfies property (1′).

Proof. This is a consequence of the facts that $\tilde{S}$ (resp., $\tilde{T}$) is a union of pairwise disjoint copies of S (resp., T) and a subset of irrationals, and that each open subset of $\tilde{S}$ (resp., $\tilde{T}$) contains a copy of S (resp., T). □

2. An application to absorbers

For a compactum K, let $\mathcal{C} = \mathcal{C}(K)$ be the class of compacta embeddable in $K \times \sigma$ (notice that the class $\mathcal{C}$ is $[0,1]$-multiplicative, i.e., for $L \in \mathcal{C}$, $L \times [0,1] \in \mathcal{C}$). There exists an absorber $\Omega(K)$ for the class $\mathcal{C}$ (see [BRZ] for the definition). We will describe $\Omega(K)$, as done in [D]. Let

$$\mathcal{E} = \{(x_i) \in \ell^2 | \sum_1^\infty i^2 x_i^2 \leq 1\}$$

be the i-d convex ellipsoid in ℓ^2, a topological copy of Q, and

$$B = \{(x_i) \in \ell^2 | \sum_i^\infty i^2 x_i^2 = 1\} \subset \mathcal{E}$$

be its pseudoboundary. Embed K into B such that $K \subset B$ is linearly independent and there exists a countable, linearly independent $D \subset B \setminus K$ dense in B. Notice that $\text{span}(D) \cap \mathcal{E}$ is a topological copy of σ (which is also denoted by σ). Define

$$\Omega(K) = \{tk + (1-t)x | k \in K, x \in \sigma, t \in [0,1]\}.$$

Most absorbers enjoy a regular structure, but absorbers of the form $\Omega(K)$ for nontypical K are themselves nontypical. Since $\Omega(K)$ is a countable union of elements of $\mathcal{C}$, applying Remark 1, we obtain:

Theorem. *For the absorber $\Omega(K)$, we have:*

(a) *if K satisfies property* (1)*, then $\Omega(K) \times \Omega(K) \not\cong \Omega(K)$;*

(b) *if K satisfies property $(1')$, then there is no f-i of $\Omega(K) \times \Omega(K)$ into $\Omega(K)$; in particular, there is no group or convex structure on $\Omega(K)$.*

Corollary. *None of the absorbers $\Omega(P)$, $\Omega(\tilde{S})$, and $\Omega(\tilde{T})$ is homeomorphic to its square. They are pairwise nonhomeomorphic. Moreover, $\Omega(P)$ and $\Omega(\tilde{T})$ do not carry a group structure or a convex structure.*

Proof. It is enough to show that

$$\omega(P) \ncong \omega(T').$$

To see this use the facts that: (1) every open subset of P contains a copy of P, (2) P is connected, (3) P contains closed subsets of all finite dimensions. As a consequence, no open subset of P can be embedded into $\tilde{T} \times I^q$. □

With an extra work (see [D]), we obtain:

Remark 2. For $n < m$,
a) $\Omega(\tilde{S})^m \ncong \Omega(\tilde{S})^n$;
b) $\Omega(P)^m$ does not admit a f-i map into $\Omega(P)^n$.

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