REFLECTIONS ON THE BING-BORSUK CONJECTURE

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The question as to whether a homogeneous euclidean neighborhood retract (ENR) is a topological manifold goes back, at least, to the paper by Bing and Borsuk [2] in which they show that an n-dimensional homogeneous ENR is a topological manifold when $n < 3$. In this paper they discuss the question as to whether the result holds in higher dimensions and suggest that, at the least, homogeneous ENR's should be generalized manifolds (i.e., ENR homology manifolds). One of the main conjectures in [6] is that a generalized n-manifold, $n \geq 5$, satisfying the disjoint disks property is homogeneous. Thus, the spaces constructed in [6] may provide examples of homogeneous ENR's that are not topological manifolds. Another possible example was constructed by Jakobsche in [11] in dimension 3, assuming the Poincaré conjecture is false. Our first attempt to show that a homogeneous ENR is a homology manifold [5] succeeded at the expense of imposing the condition that the local homology groups of the space are finitely generated in all dimensions. This result was, in fact, already to be found in [4]. More specifically, the following theorem is known:

Theorem 1 ([4, 5]). *If X is an n-dimensional, homogeneous* ENR*, and $H_k(X, X - x; \mathbb{Z})$ is finitely generated for some* (*and, hence, all*) *x, then X is a homology manifold.*

In this talk we discuss attempts to prove the conjecture of Bing and Borsuk:

Conjecture 1. *If X is an n-dimensional, homogeneous* ENR*, then X is a homology n-manifold.*

Related to this conjecture is an older conjecture of Borsuk [3].

Conjecture 2. *There is no finite dimensional, compact, absolute retract.*

Definitions. A **homology n-manifold** is a space X having the property that for each $x \in X$,

Key words and phrases. generalized manifolds, homogeneity.

$$H_k(X, X - x; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n. \end{cases}$$

A **euclidean neighborhood retract** (ENR) is a space homeomorphic to a closed subset of euclidean space that is a retract of some neighborhood of itself. A topological space X is **homogeneous** if, for any two points x and y in X, there is a homeomorphism of X onto itself taking x to y.

We will assume from now on that X is a n-dimensional homogeneous ENR and R is a PID. It's easy to get started:

Lemma 1. *For all $x \in X$, $H_0(X, X - x; R) = 0$, if $n > 0$ and $H_1(X, X - x; R) = 0$, if $n > 1$.*

One of the main problems that arises is the possibility that for some (and hence, all) $x \in X$, $H_k(X, X - x; \mathbb{Z})$ is infinitely generated for some $k \geq 2$. This difficulty could be overcome for $k < n$, if k-dimensional homology classes are carried by k-dimensional subsets of X. There are counterexamples for k-dimensional *homotopy* classes when $k \geq 2$ [7, 10], but I know of no counterexamples for carriers of homology classes.

Via Alexander duality, mapping cylinder neighborhoods provide an alternative way to view the local homology groups of X. Assume X is nicely embedded in $\mathbb{R}^{n+m}$, for some $m \geq 3$, so that X has a mapping cylinder neighborhood $N = C_\phi$ of a map $\phi \colon \partial N \to X$, with mapping cylinder projection $\pi \colon N \to X$ [12, 13]. Given a subset $A \subseteq X$, let $A^* = \pi^{-1}(A)$ and $\dot{A} = \phi^{-1}(A)$.

By a result of Daverman-Husch [8], the Bing-Borsuk Conjecture is equivalent to

Conjecture 3. *$\pi \colon N \to X$ is an approximate fibration.*

Duality shows that the local homology of X is captured in the cohomology of the fibers of this map (in the dual dimensions).

Lemma 2. *If A is a closed subset of X, then $H_k(X, X - A; R) \cong \check{H}_c^{n+m-k}(A^*, \dot{A}; R)$.*

Proof. Suppose A is closed in X. Since $\pi \colon N \to X$ is a proper homotopy equivalence,

$$H_k(X, X - A; R) \cong H_k(N, N - A^*; R).$$

Since ∂N is collared in N,

$$H_k(N, N - A^*; R) \cong H_k(\mathrm{int} N, \mathrm{int} N - A^*; R),$$

and by Alexander duality,

$$H_k(\mathrm{int}N, \mathrm{int}N - A^*; R) \cong \check{H}_c^{n+m-k}(A^* - \dot{A}; R)$$

$$\cong \check{H}_c^{n+m-k}(A^*, \dot{A}; R)$$

(since $\dot{A}$ is also collared in A^*). □

Lemma 3. $H_k(X, X - x; R) = \varinjlim H_k^{\ell f}(U; R)$, *where the limit is taken over open neighborhoods* U *of* x.

Proof. Again, using Lemma 2 and the fact that π is proper, we have, for each neighborhood U of x in X,

$$H_k^{\ell f}(U; R) \cong H_k^{\ell f}(U^*; R) \cong$$

$$H^{n+m-k}(U^*, \dot{U}; R) \to \check{H}^{n+m-k}(x^*, \dot{x}; R) \cong H_k(X, X - x; R).$$

□

As the next lemma shows, homogeniety, specifically microhomogeneity, implies that any finitely generated submodule of the local homology module $H_k(X, X - x; R)$ propagates naturally to all points near x.

Lemma 4. *Suppose* F *is a finitely generated submodule of* $H_k(X, X - x; R)$, $k \geq 0$. *Then there is a neighborhood* U *of* x *and a submodule* $F_0 \subseteq H_k(X, X - U; R)$ *such that*

(i) $F_0 = \mathrm{im}\, F$ *under inclusion,*
(ii) *for all* $y \in U$, *the inclusion* $H_k(X, X - U; R) \to H_k(X, X - y; R)$ *is one-to-one on* F_0.

Proof. Given finitely generated $F \subseteq H_k(X, X - x; R)$.

Let $a_1, \ldots, a_r$ be generators of F, represented by singular chains $c_1, \ldots, c_r$, respectively, and let $B_1, \ldots, B_r$ be the carriers of $\partial c_1, \ldots, \partial c_r$, respectively. $B_1 \cup \ldots \cup B_r$ is a compact set in $X - x$, and there is a neighborhood U_1 of x such that for every smaller neighborhood V of x,

$$F \subseteq \mathrm{im}(H_k(X, X - V; R) \to H_k(X, X - x; R).$$

By Effros Theorem [9, 1], homogeneity implies micro-homogeneity: Given $\epsilon > 0$ there is a $\delta > 0$, such that if $d(x, y) < \delta$, then there is a homeomorphism $h_y \colon X \to X$ such that $h_y(x) = y$ and h_y moves no point of $(B_1 \cup \ldots \cup B_r)$ more than ϵ.

For ϵ small, h_y is homotopic to the identity on X by a homotopy whose restriction to $(B_1 \cup \ldots \cup B_r)$ has image in $X - x$, hence, in $X - U$ for some neighborhood U of x. □

The Leray spectral sequence of the Leray sheaf $\mathcal{H}^q(\pi)$ of $\pi\colon N \to X$, with stalk $\mathcal{H}^q(\pi)_x = \check{H}^q(x^*, \dot{x}; R)$, has E_2-term

$$E_2^{p,q} = H_c^p(X; \mathcal{H}^q(f)),$$

and converges to

$$E_\infty^{p,q} = H_c^{p+q}(N, \partial N; R).$$

In [5] it is proved that the Bing-Borsuk Conjecture is equivalent to

Conjecture 4. *For all q, $\mathcal{H}^q(\pi)$ is locally constant.*

Theorem 2. *If R is a PID, then $H_n(X, X - x; R) \neq 0$. Moreover, if U is a sufficiently small neighborhood of x, $H_c^n(U; R) \neq 0$, and $H_n^{\ell f}(U; R) \neq 0$ and free.*

Proof. Since U is an ENR of dimension n, the locally finite homology of U can be computed from a chain complex (using nerves of sufficiently fine covers of U of order $n + 1$) that is 0 in dimension $n + 1$; hence, $H_n^{\ell f}(U; R)$ is free. Thus, $H_c^n(U; R) = 0$ implies $H_n^{\ell f}(U; R) = 0$. If $H_n^{\ell f}(U; R) = 0$ for every neighborhood U of x, then $\check{H}^m(x^*, \dot{x}; R) \cong H_n(X, X - x; R) = \varinjlim H_n^{\ell f}(U; R) = 0$, so that $\mathcal{H}^m$ is the 0 sheaf.

Restrict the map π to $(U^*, \dot{U})$, where U is an open neighborhood of x. By definition,

$$E_3^{n,q} = \ker(d_2\colon E_2^{n,q} \to E_2^{n+2,q-1})/\operatorname{im}(d_2\colon E_2^{n-2,q+1} \to E_2^{n,q}).$$

Since $\dim U = n$ implies $E_2^{n+2,q-1} = 0$, so that $E_2^{n,m}$ maps onto $E_3^{n,m}$. Similarly, $E_r^{n,m}$ maps onto $E_{r+1}^{n,m}$, for $r \geq 2$, so that $E_2^{n,m}$ maps onto $E_\infty^{n,m}$. However, if U is connected, $E_\infty^{n,m} = H_c^{n+m}(U^*, \dot{U}; R) \cong R \neq 0$. Hence, $\mathcal{H}^m$ is not 0, which, in turn, implies $H_n^{\ell f}(U; R) \neq 0$ and $H_c^n(U; R) \neq 0$, for some neighborhood U of x. □

Remark. The argument in this proof can be used to see that $H_c^n(X; \mathcal{H}^m) \neq 0$; but, if $H_n(X, X - x; R)$ is not finitely generated, we cannot necessarily conclude that the ordinary cohomology of X is nonzero. If so, we would have a proof of Conjecture 2.

Suppose that F is a finitely generated submodule of $H_k(X, X - x; R)$. By Lemma 4 there is a neighborhood U of x and a constant sheaf $\mathcal{F}$ on U such that $\mathcal{F} \subseteq \mathcal{H}^q|U$, $q = n + m - k$, and $\mathcal{F}_x = F$. Since $\dim U = n$, the short exact sequence of sheaves

$$0 \to \mathcal{F} \to \mathcal{H}^q|U \to \operatorname{coker} \iota \to 0$$

induces a long exact sequence on Borel-Moore homology

$$0 \to H_n(U; \mathcal{F}) \to H_n(U; \mathcal{H}^q)$$
$$\to H_n(U; \operatorname{coker} \iota) \to \cdots ,$$

which implies $H_n(U;\mathcal{F}) \to H_n(U;\mathcal{H}^q)$ is one-to-one.

We would like for the same to be true for inclusion in cohomology,

$$\operatorname{im}(H^n(U;\mathcal{F}) \to H^n(U;\mathcal{H}^q)),$$

since this would allow us to get a good relationship between sheaf cohomology of U and ordinary cohomology of $(U^*, \dot{U})$.

Unfortunately, there is nothing that seems to preclude the Bockstein

$$H^{n-1}(U;\operatorname{coker}\iota) \to H^n(U;\mathcal{F})$$

from being onto. Indeed, it is possible to construct a rather "homogeneous" looking sheaf over the interval $(0, 1)$, having infinitely generated stalks, for which this Bockstein (with $n = 1$) is onto.

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