TOPOLOGICAL SINGULARITIES IN COSMOLOGY

LAWRENCE BRENTON

Introduction. By definition a space-time is a smooth four-dimensional manifold X admitting a Lorentzian metric g whose curvature tensors satisfy the Einstein field equations for some "reasonable" distribution of matter and energy.

On the other hand, the singularity theorems of Penrose and Hawking [8] assert that any such space-time must contain singular points. In other words, it can't be a smooth manifold with metric g defined at every point.

I suppose that the logical conclusion is that the universe cannot exist. Yet somehow God was able to overcome this difficulty [5]. Can we?

The best-publicized attempt is due to Hawking and Hartle [7]. The Hawking-Hartle "no boundary" theory has been popularized as a theory of "imaginary time." A better description of the geometry of the model is given by saying that in a small neighborhood of the "big bang" the metric changes signature, becoming positive definite locally. The physical interpretation is that at the moment of creation none of the four directions in space-time had yet been distinguished as "time."

The original model for this construction was the closed positively curved model characterized physically by the condition that the total mass-energy content of the cosmos is greater than the "critical value" ($\Omega > 1$). Although the no boundary concept has been extended to the flat and negatively curved standard models now favored by extragalactic observations [9], the positively curved model remains the most successful in revealing the geometry of space-time near the big bang singularity.

The reason for this is clear. In the standard closed model the spacelike cross sections (all of space at a particular moment in time) are three-spheres. The entirety of space-time up to the present is viewed as an expanding family of such spheres originating in a "sphere of radius zero" at the moment of creation. Topologically, that is to say, the history of the cosmos so far is the cone on S^3. Since the cone on S^3 is just the ordinary four-disc, topologically there is nothing to distinguish the moment of the big bang from any other point of space-time. Thus the singularity at the beginning of time, whose existence is guaranteed by the Penrose-Hawking theorems, is "merely" geometrical and physical, not topological. Curvature tensors associated with the metric diverge to infinity, as does the mass-energy density. But the background structure of space-time maintains its integrity as a topological manifold even at the singular point.

Alas, the real world intrudes into our theorizing. Beginning in 1998 data from deep space studies using space-based telescopes and large array imaging techniques have effectively ruled out the positively curved model in favor of negatively curved models and (most popular currently) flat models with substantial cosmological constant (dark energy) [15]. At the same time there has been an explosion of interest in cosmological models whose space-like cross sections are not simply connected [11]. Indeed, physicists have not presented any reasons for preferring simply connected models except a vague feeling that such models are "simpler" than multiply connected ones.

In this note we present a family of topological spaces in which the requirement of simple connectivity is weakened to the condition that the "space-like" submanifolds are homology 3-spheres. These spaces have the feature that their geometric properties are underlain by exotic topological structure at the singular point.

The construction. Let $S_1, \ldots, S_k$ be a collection of 2-spheres, let Γ be an acyclic graph on vertices $v_1, \ldots, v_k$, and let $w_1, \ldots, w_k$ be integer "weights" assigned to the vertices. Denote by $E_i, i = 1, \ldots, k$, the total space of the 2-plane bundle on S_i with Euler number $-w_i$. Plumb these spaces together according to the prescription of the graph Γ. That is, locally identify the zero section of E_i with a fiber of E_j, and vice versa, whenever v_i meets v_j in Γ. Let M denote the compact three-manifold obtained by taking the union of the plumbed unit circle bundles of the E_i's, and smoothing the corners. Finally, let X be the space obtained by collapsing the zero sections to a point P. The resulting space X is homeomorphic to the cone on M, and is a smooth four-manifold except at the singular point P.

Theorem ([2])**.** *For $\Gamma, w_1, \ldots, w_k$ as above, denote by $A(\Gamma)$ the "dual intersection matrix"* $\mathrm{diag}(w_1, \ldots, w_k)$ *— adjacency matrix of Γ. Then the 3-fold M of the construction is a homology 3-sphere if and only if the determinant of $A(\Gamma) = \pm 1$.*

Moreover, if $A(\Gamma)$ is positive definite, then X admits the structure of a two-(complex)-dimensional complex algebraic variety, with a unique singular point at the origin [6]. Since the germ of the variety at the singular point determines the topology of the entire space, such singular complex surfaces provide an interesting setting in which to study the relations between the topology and the geometry, hence the physics, of big bang models in cosmology, with the compact homology 3-spheres M playing the role of the space-like submanifolds in space-time. The central question motivating this inquiry is this:

Guiding question: To what extent are the geometrical and physical properties of big bang space-time models determined by the topology of the singular point?

Egyptian fractions. One way to obtain particular examples of such spaces is as follows.

Theorem ([3]). *Let $n_1, \ldots, n_k$ be a solution in positive integers to one of the two unit fraction Diophantine equations*

$$(*) \qquad \Sigma 1/n_i = 1 \pm 1/\Pi n_i.$$

In the "minus" case, let Γ be the star graph with a central vertex of weight $w_0 = 1$, and with k arms of length 1, with weight $w_i = n_i$ on the single vertex of the ith arm. In the "plus" case, we take Γ to be the star-shaped graph whose central vertex has weight $w_0 = k-1$ and whose ith arm consists of $k-1$ vertices, each of weight 2. Then the 3-fold $M = M(\Gamma, n_1, \ldots, n_k)$ of the construction outlined above is a homology 3-sphere.

This raises a question in number theory, which is interesting in its own right and which enjoys a distinguished history dating back 4000 years to dynastic Egypt [4, 16, 17]: For fixed k, find all solutions in positive integers $n_1, \ldots, n_k$ to the equations $(*)$. Not only is this a fun and instructive problem, but also it is one that undergraduate students can understand and tackle. With motivation from the geometry of complex surfaces and the possible relevance of this topic to cosmological models, the Wayne State Undergraduate Research Group ("Surge" – the W in the acronym is silent) has attacked this problem with great vigor. After several semesters of work by of a total of 33 students involved in the program, the students, much to my pride and joy, succeeded in producing the complete list of all solutions through $k = 8$. There are 160 solutions to the minus equation and 598 solutions to the plus equation in this range [10, 12].

Examples. The most intensively studied example is the equation

$$1/2 + 1/3 + 1/5 = 1 + 1/30.$$

The corresponding weighted graph Γ is the Dynkin diagram (Coxeter graph) E_8 of the root system of the simple complex Lie algebra e_8.

The associated complex surface singularity is the rational double point given in complex co-ordinates x, y, z on $\mathbb{C}^3$ as the zero set

(a) $\{x^2 + y^3 + z^5 = 0\} \subset \mathbb{C}^3$

The compact 3-fold M of the construction of this paper for this weighted graph is homeomorphic to the Poincaré 120-cell [14]. The first homotopy group is the group of rigid motions of the dodecahedron. This is a well-known finite perfect group of order 120; its Abelianization is trivial, hence M is a homology 3-sphere as required.

The simply connected covering space in this example is S^3. In fact, M is obtained by a tiling of S^3 by 120 "twisted" dodecahedra. Thus M inherits a homogeneous, isotropic line element $d\sigma$ of constant positive curvature from S^3. As above, let X denote the cone on M, and define a metric on X by $ds^2 = -t^{4/3}dt^2 + d\sigma^2$. The resultant space-time model, via the Einstein field equations, satisfies the physical requirements of spatially homogeneous distribution of matter, decreasing in density proportionally to t^{-2} from an infinitely dense big bang singularity. This model is indistinguishable locally from the matter-dominated "dust" model in standard cohomology. In principle its validity could be verified by the discovery of "ghosts"—multiple sightings of the same galaxy cluster in different directions—or by analyses of distinctive patterns of inhomogeneities in the cosmic microwave background radiation. Serious experiments are underway by astronomers seeking to detect just such heavenly anomalies (mostly working in the context of the 3-torus model), but so far without success [20].

In [1] I gave the details of a similar treatment of the complex hypersurface

(b) $\{x^2 + y^3 + z^6 = 0\} \subset \mathbb{C}^3$.

Since 1, 2, and 6, do not satisfy either of the relations $(*)$ we do not obtain a homology 3-sphere by the construction of this paper. However, if we intersect this complex variety X with a 5-sphere in $\mathbb{C}^3$, the intersection is a smooth compact 3-fold M and X is locally the cone on M [13]. Thus the topological space X is a candidate for a big bang space-time model.

This 3-fold M turns out to be homeomorpic to a non-trivial S^1-bundle on the 2-torus, with $H_1(M, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$. Furthermore, M admits naturally a metric that extends to a Robertson-Walker metric on the cone X and which is homogeneous on the space-like cross sections [18]. The metric is not, however, fully isotropic; the physical result is a tiny amount of universal pressure in the direction of the fiber of M, regarded as a circle bundle on T^2. See [1] for the details of the geometry and physical interpretation of this model.) This space-time begins in a singular point of infinite density and pressure, expands to a maximum size, and then contracts symmetrically to a "big crunch." Indeed, the "size" $R(t)$ of the universe at time t is given by the inverse relation

$$(**) \qquad t/(2C) = \arcsin \sqrt{R/C} - \sqrt{(R/C)(1 - R/C)}$$

where C is a constant of integration representing the maximum size of the universe at the end of the expansion phase.

Open question: Are these physical properties of the model determined by the topology at the singular point, or do they vary with choice of metric?

To complete this cycle of ideas, consider the complex variety

(c) $\{x^2 + y^3 + z^7 = 0\} \subset \mathbb{C}^3$.

Since $1/2 + 1/3 + 1/7 = 1 - 1/(2 * 3 * 7)$ (the "minus" version of equation $(*)$), we obtain a very inviting topological space X, the cone on a homology 3-sphere M, whose singularity at the origin is very well understood by algebraic geometers [20]. The fundamental group is presented by generators $\alpha, \beta, \gamma, \omega$, with relations $\alpha^2 = \beta^3 = \gamma^7 = \alpha\beta\gamma = \omega$. This group is an infinite perfect group that is a non-trivial central extension by $\mathbb{Z}$ of the group of symmetries of the tiling of the Poincaré disc by triangles with angles $\pi/2, \pi/3$, and $\pi/7$.

Open question: Does there exist a homogeneous Lorentzian metric on this singular space-time candidate, which exhibits realistic physical properties?

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Department of Mathematics, Wayne State University, Detroit, Michigan 48202, USA