Math 571: Advanced Ordinary Differential Equations

Prof. Todd Kapitula *
Department of Mathematics and Statistics
University of New Mexico

December 3, 2004

*E-mail: kapitula@math.unm.edu

Contents

1. Example: Bose-Einstein Condensates

A more complete description of the underlying physics, as well as the mathematical formulation, can be found in [**18**, **19**, **21**]. Upon a suitable rescaling, the quasi-one dimensional model for a Bose-Einstein condensate with both a magnetic trap and optical lattice is given by

$$\mathrm{i}q_t = \mathcal{L}q - \omega q + \epsilon p(t)\cos(\frac{2x}{\mu})q - \delta|q|^2 q. \tag{1.1}$$

Here $q : \mathbb{R}\times\mathbb{R}^+ \mapsto \mathbb{C}$,

$$\mathcal{L} := -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2 + V_0\cos(\frac{2x}{\mu}),$$

$p : \mathbb{R} \mapsto \mathbb{R}$ is smooth and satisfies $p(t+T) = p(t)$, $\epsilon, \mu \in \mathbb{R}^+$, and $\delta \in \{-1, +1\}$. As discussed in [**18**, Section 2], one knows the following about $\mathcal{L}$:

(a) $\sigma(\mathcal{L}) = \{\mu_j\}_{j=0}^{\infty} \subset \mathbb{R}$, and each eigenvalue is simple with $\mu_j < \mu_{j+1}$ for all $j \in \mathbb{N}_0$

(b) the eigenfunctions $\{q_j(x)\}_{j=0}^{\infty}$ form a complete orthonormal basis, and satisfy the decay condition

$$|q_j(x)|\mathrm{e}^{x^2/8} \le C, \quad x \in \mathbb{R}$$

(c) if j is even, the eigenfunction is even, whereas if j is odd, the eigenfunction is odd.

The desire is to construct a low-dimensional approximation to equation (1.1). Upon writing

$$q(x,t) \sim c_0(t)q_0(x) + c_1(t)q_1(x),$$

where $c_j : \mathbb{R} \mapsto \mathbb{C}$, and performing several rescalings, one sees that the ODE governing the behavior of $c_j(t)$ is given by

$$\mathrm{i}\dot{c}_j = \frac{\partial\mathcal{H}}{\partial\bar{c}_j}, \quad j = 0, 1, \tag{1.2}$$

where $\mathcal{H}$ is a particular Hamiltonian (see [**19**, Section 2] for the details). One feature of equation (1.2) is that

$$|c_0(t)|^2 + |c_1(t)|^2 \equiv 2, \tag{1.3}$$

which is a consequence of the fact that equation (1.1) conserves the total number of atoms. Upon setting $c_j := \sqrt{2\rho_j}\,\mathrm{e}^{\mathrm{i}\phi_j}$, i.e., the pairs (ρ_j, ϕ_j) define action-angle variables, one finds that

$$\dot{\rho}_j = \frac{\partial\mathcal{H}}{\partial\phi_j}, \quad \dot{\phi}_j = -\frac{\partial\mathcal{H}}{\partial\rho_j}; \qquad j = 0, 1, \tag{1.4}$$

where now

$$\begin{aligned}\mathcal{H}(\rho_0,\rho_1,\phi_0,\phi_1,t) = {} & \frac{1}{2}(\mu_0-\omega)\rho_0 + \frac{1}{2}(\mu_1-\omega)\rho_1 + \epsilon p(t)(\alpha_{00}^{\mathrm{p}}\rho_0 + \alpha_{11}^{\mathrm{p}}\rho_1) \\ & - \delta R^2\left(\frac{1}{2}\alpha_{000}^0\rho_0^2 + \frac{1}{2}\alpha_{111}^1\rho_1^2 + \alpha_{011}^0(2+\cos 2\Delta\phi)\rho_0\rho_1\right).\end{aligned}$$

Here $\Delta\phi := \phi_1 - \phi_0$, and R is a measure of the total number of atoms in the condensate. Furthermore, the coefficients are given by

$$\alpha_{ijk}^m := \langle q_i q_j, q_k q_m\rangle, \quad \alpha_{ij}^{\mathrm{p}} := \langle\cos(\frac{2x}{\mu})q_i, q_j\rangle.$$

Since equation (1.3) implies that $\rho_0 + \rho_1 \equiv 1$, after some algebra equation (1.4) can be reduced to

$$\begin{aligned}\dot{\rho}_0 &= -2\delta R^2\rho_0(1-\rho_0)\sin 2\Delta\phi \\ \dot{\Delta\phi} &= -\frac{1}{2}(\mu_1-\mu_0) - \epsilon(\alpha_{11}^{\mathrm{p}} - \alpha_{00}^{\mathrm{p}})p(t) \\ &\quad + \delta R^2\left(\alpha_{111}^1 - (\alpha_{000}^0 + \alpha_{111}^1)\rho_0 + \alpha_{011}^0(2+\cos 2\Delta\phi)(2\rho_0 - 1)\right).\end{aligned} \tag{1.5}$$

Note that equation (1.5) can be considered on the cylinder $(\rho_0, \Delta\phi) \in [0,1] \times [0, 2\pi)$. Further note that if $\epsilon = 0$ the system is autonomous; hence, it can be considered to be a periodic perturbation of a planar system. Systems of such type are considered in detail in [9], and will be discussed herein.

Remark 1.1. In class I will present some numerical simulations of equation (1.5) with $p(t) = \cos t$ using the software Dynamics Solver. The coefficients were derived upon setting $\mu = \sqrt{.2}$ and $V_0 = 2.5$.

2. Invariant Manifolds

For the system

$$\dot{x} = f(x), \quad x(0) = x_0,$$

the solution will often be represented as $\phi_t(x_0)$. The function ϕ_t defines a flow, i.e.,

(a) $\phi_{s+t} = \phi_s \phi_t = \phi_t \phi_s$

(b) $\phi_0 = \mathbb{1}$

2.1. Linear systems

First consider

$$\dot{x} = Ax, \quad x(0) = x_0,$$

where $A \in \mathbb{R}^{n \times n}$. In this case it is known that

$$\phi_t(x) = \mathrm{e}^{At} x.$$

Let $\gamma_{\mathrm{s}} < \gamma_{\mathrm{u}} \in \mathbb{R} \backslash \sigma(A)$ be given, and denote

$$\begin{aligned} \sigma^{\mathrm{s}}(A) &:= \{\lambda \in \sigma(A) \,:\, \mathrm{Re}\,\lambda < \gamma_{\mathrm{s}}\} \\ \sigma^{\mathrm{c}}(A) &:= \{\lambda \in \sigma(A) \,:\, \gamma_{\mathrm{s}} < \mathrm{Re}\,\lambda < \gamma_{\mathrm{u}}\} \\ \sigma^{\mathrm{u}}(A) &:= \{\lambda \in \sigma(A) \,:\, \gamma_{\mathrm{u}} < \mathrm{Re}\,\lambda\}. \end{aligned}$$

If $\gamma_{\mathrm{s,u}}$ are chosen so that $\lambda \in \sigma^{\mathrm{c}}(A)$ if and only if $\mathrm{Re}\,\lambda = 0$, then one has the usual definition of these spectral subsets. Associated with each spectral set there is a subspace $E^{\mathrm{s,c,u}}$ which satisfies the property that

$$\mathrm{e}^{At} E^{\mathrm{s,c,u}} \subset E^{\mathrm{s,c,u}}.$$

Furthermore, one can easily prove the following lemma.

Lemma 2.1. *If $x_0 \in E^{\mathrm{s}}$, then there exists a constant $M \geq 1$ such that*

$$|\mathrm{e}^{At} x_0| \mathrm{e}^{-\gamma_{\mathrm{s}} t} \leq M, \quad t \geq 0.$$

If $x_0 \in E^{\mathrm{u}}$, then there exists a constant $M \geq 1$ such that

$$|\mathrm{e}^{At} x_0| \mathrm{e}^{-\gamma_{\mathrm{u}} t} \leq M, \quad t \leq 0.$$

Finally, if $x_0 \in E^{\mathrm{c}}$, then there exists a constant $M \geq 1$ such that

$$|\mathrm{e}^{At} x_0| \mathrm{e}^{-\gamma_{\mathrm{u}} t} \leq M, \quad t \geq 0 \qquad \text{and} \qquad |\mathrm{e}^{At} x_0| \mathrm{e}^{-\gamma_{\mathrm{s}} t} \leq M, \quad t \leq 0.$$

Remark 2.2. With the usual definition, i.e., upon setting $\gamma_{\mathrm{s}} = \gamma_{\mathrm{u}} = \epsilon$ with $\epsilon > 0$ begin arbitrarily small, one can paraphrase Lemma 2.1 to state that if $x_0 \in E^{\mathrm{s}}$, then $\phi_t(x_0) \to 0$ exponentially fast as $t \to +\infty$, while if $x_0 \in E^{\mathrm{u}}$, then $\phi_t(x_0) \to 0$ exponentially fast as $t \to -\infty$. If $x_0 \in E^{\mathrm{c}}$, then the behavior has no a-priori characterization.

2.2. *The manifold theorems*

Assume that for the ODE $\dot{x} = f(x)$ one has that $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ is C^r ($r \geq 2$) with $f(0) = 0$.

Definition 2.3. A space X is a topological manifold of dimension k if each point $x \in X$ has a neighborhood homeomorphic to the unit ball in $\mathbb{R}^k$.

Remark 2.4. In particular, the graphs of smooth functions are manifolds.

Definition 2.5. Let N be a given small neighborhood of $x = 0$, and let $\epsilon > 0$ be sufficiently small. The stable manifold, W^{s}, is

$$W^{\mathrm{s}} := \{x \in N : \phi_t(x)\mathrm{e}^{-\gamma_{\mathrm{s}} t} \in N \,\forall\, t \geq 0 \text{ and } \phi_t(x)\mathrm{e}^{-\gamma_{\mathrm{s}} t} \to 0 \text{ exponentially fast as } t \to +\infty\}.$$

The unstable manifold, W^{u}, is

$$W^{\mathrm{u}} := \{x \in N : \phi_t(x)\mathrm{e}^{-\gamma_{\mathrm{u}} t} \in N \,\forall\, t \leq 0 \text{ and } \phi_t(x)\mathrm{e}^{-\gamma_{\mathrm{u}} t} \to 0 \text{ exponentially fast as } t \to -\infty\}.$$

The center manifold, W^{c}, is invariant relative to N, i.e., if $x \in W^{\mathrm{c}}$, then

$$\phi_t(x)\mathrm{e}^{-\gamma_{\mathrm{u}} t} \in W^{\mathrm{c}} \cap N, \quad t \geq 0 \qquad \text{and} \qquad \phi_t(x)\mathrm{e}^{-\gamma_{\mathrm{s}} t} \in W^{\mathrm{c}} \cap N, \quad t \leq 0.$$

Furthermore, $W^{\mathrm{c}} \cap W^{\mathrm{s}} = W^{\mathrm{c}} \cap W^{\mathrm{u}} = \{0\}$.

Theorem 2.6 (Stable manifold theorem)**.** *There is a neighborhood N of $x = 0$ and a C^{r-1} function $h^{\mathrm{s}} : N \cap E^{\mathrm{s}} \mapsto E^{\mathrm{c}} \oplus E^{\mathrm{u}}$ such that $W^s = \mathrm{graph}(h^{\mathrm{s}})$.*

Theorem 2.7 (Unstable manifold theorem)**.** *There is a neighborhood N of $x = 0$ and a C^{r-1} function $h^{\mathrm{u}} : N \cap E^{\mathrm{u}} \mapsto E^{\mathrm{s}} \oplus E^{\mathrm{c}}$ such that $W^u = \mathrm{graph}(h^{\mathrm{u}})$.*

Theorem 2.8 (Center manifold theorem)**.** *There is a neighborhood N of $x = 0$ and a C^{r-1} function $h^{\mathrm{c}} : N \cap E^{\mathrm{c}} \mapsto E^{\mathrm{u}} \oplus E^{\mathrm{s}}$ such that $\mathrm{graph}(h^{\mathrm{u}})$ is a W^{c}.*

Remark 2.9. One has that:

(a) $\dim(W^{\mathrm{s,c,u}}) = \dim(E^{\mathrm{s,c,u}})$

(b) The manifolds are invariant, i.e., if $x \in W^{\mathrm{s,c,u}}$, then $\phi_t(x) \in W^{\mathrm{s,c,u}}$ for all $t \in \mathbb{R}$. Thus, if, e.g., $\dim(E^{\mathrm{s}}) = k_{\mathrm{s}}$, there is then a k_{s}-dimensional ODE which governs the behavior of the flow on W^{s}.

(c) $W^{\mathrm{s,c,u}}$ is tangent to $E^{\mathrm{s,c,u}}$ at $x = 0$.

(d) The dynamical behavior on W^{s} and W^{u} is determined solely by the linear behavior.

(e) W^{c} is not unique. For example, consider the system

$$\dot{x} = x^2, \quad \dot{y} = -y.$$

(f) The proofs of these theorems is discussed in [28, Chapter 3.3.5].

The formulation of the manifold theorems has important implications. First, one can foliate W^{u}, as well as W^{s}, in the following manner. Let $\lambda_1, \dots, \lambda_k \in \sigma(\mathrm{D}f(0,0))$ be simple, and hence real, be such that $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k$, and let the associated eigenvector be denoted by v_j. By the statement of the unstable manifold theorem it is known that there is an invariant k-dimensional manifold W^{u} which is tangent to

$$E^{\mathrm{u}} = \mathrm{span}\{v_1, \dots, v_k\}$$

at $x = 0$. For a given $1 \leq \ell < k$ let γ_{u} be chosen so that $\mathrm{Re}\,\lambda_\ell < \gamma_{\mathrm{u}} < \mathrm{Re}\,\lambda_{\ell+1}$. From the statement of the unstable manifold theorem there is an invariant manifold of dimension $k - \ell$, say W^{uu}_ℓ, which is tangent at $x = 0$ to

$$E^{\mathrm{uu}}_\ell := \mathrm{span}\{v_{\ell+1}, \dots, v_k\}.$$

Note that $W^{\mathrm{uu}}_\ell \subset W^{\mathrm{u}}$. In this way one can foliate W^{u} with

$$W^{\mathrm{uu}}_{k-1} \subset W^{\mathrm{uu}}_{k-2} \subset \cdots \subset W^{\mathrm{uu}}_1 \subset W^{\mathrm{u}}.$$

Now consider the system

$$\dot{x} = f(x, \lambda), \quad \dot{\lambda} = 0,$$

where $f(0,0) = 0$ and $\lambda \in \mathbb{R}^k$. Suppose that γ_{s} and γ_{u} are chosen so that

$$\sigma^{\mathrm{c}}(\mathrm{D}f(0,0)) = \{\lambda \in \sigma(\mathrm{D}f(0,0)) \,:\, \mathrm{Re}\,\lambda = 0\},$$

and further suppose that $\dim(E^{\mathrm{c}}) = \ell$. As an application of the center manifold theorem there then exists an invariant $(k+\ell)$-dimensional invariant manifold, so that the governing equations are of dimension $k+\ell$. This reduced set of equations will determine all of the interesting bifurcations, as it is known that any solutions on $W^{\mathrm{s}}\,(W^{\mathrm{u}})$ will have exponential behavior as $t \to +\infty\,(t \to -\infty)$.

Remark 2.10. Unless specifically told otherwise, it will henceforth be assumed that $\gamma_{\mathrm{s,u}}$ are chosen so that

$$\sigma^{\mathrm{c}}(\mathrm{D}f(0)) = \{\lambda \in \sigma(\mathrm{D}f(0)) \,:\, \mathrm{Re}\,\lambda = 0\}.$$

2.3. Examples

The manifold theorems are also important in that they implicitly tell us how to compute the flow on W^{c}. Consider the system

$$\dot{y} = Ay + g_1(y,z), \quad \dot{z} = Bz + g_2(y,z),$$

where $y \in E^{\mathrm{s}} \oplus E^{\mathrm{u}}$, $z \in E^{\mathrm{c}}$, and $|g_j(y,z)| = \mathcal{O}(|y^2| + |z^2|)$. As a consequence of the center manifold theorem it is known that W^{c} is given by the graph $y = h^{\mathrm{c}}(z)$, where $\mathrm{R}(\mathrm{D}h^{\mathrm{c}}(0)) = E^{\mathrm{c}}$. Since $h^{\mathrm{c}}(z)$ is smooth, it can be computed via a Taylor expansion. The flow on W^{c} is then given by

$$\dot{z} = Bz + g_2(h^{\mathrm{c}}(z), z),$$

as long as the trajectory stays in N.

Before considering the first example, we need the following proposition.

Proposition 2.11. *Suppose that $g : N \subset \mathbb{R}^n \mapsto \mathbb{R}^n$ is C^2 in a neighborhood of $x = 0$, and suppose that $g(0, x_2, \dots, x_n) = 0$ for all $(0, x_2, \dots, x_n) \subset N$. There exists a neighborhood $M \subset N$ and a $g_1 \in C^1(M)$ such that $g(x) = x_1 g_1(x)$.*

Proof: By Taylor's theorem one has that

$$g(x) = g(0, x_2, \dots, x_n) + \int_0^1 \frac{\partial}{\partial t} g(tx_1, x_2, \dots, x_n)\,\mathrm{d}t.$$

Choose M so that the line between x and $(0, x_2, \dots, x_n)$ lies in M. This yields that

$$\frac{\partial}{\partial t} g(tx_1, x_2, \dots, x_n) = x_1 \mathrm{D}g(tx_1, x_2, \dots, x_n)\mathbf{e}_1,$$

which then implies that

$$g_1(x) = \int_0^1 \mathrm{D}g(tx_1, x_2, \dots, x_n)\mathbf{e}_1\,\mathrm{d}t \in C^1(M). \qquad \square$$

Remark 2.12. In the next few sections we will go through a series of examples which will illuminate the power of the theory. The interested student should consult [1] for a more complete discussion.

2.3.1. Pitchfork bifurcation

For the first example, consider the system

$$\begin{aligned}
\dot{x} &= -x + x^2 - y^2 \\
\dot{y} &= \epsilon y + xy - y^3 \\
\dot{\epsilon} &= 0.
\end{aligned}$$

The equation for ϵ is appended to the system so that one can readily apply the manifold theorems. At the critical point $(0,0,0)$ one has that

$$\sigma^{\mathrm{s}}(\mathrm{D}f(0)) = \{-1\},\ E^{\mathrm{s}} = \mathrm{span}\{\mathbf{e}_1\};\quad \sigma^{\mathrm{c}}(\mathrm{D}f(0)) = \{0\},\ E^{\mathrm{c}} = \mathrm{span}\{\mathbf{e}_2, \mathbf{e}_3\}.$$

Thus, upon applying the Center Manifold Theorem one knows that the center manifold is locally given by

$$x = h(y,\epsilon);\quad h(0,0) = h_y(0,0) = h_\epsilon(0,0) = 0, \tag{2.1}$$

and the flow on W^{c} is given by

$$\begin{aligned}\dot{y} &= \epsilon y + yh(y,\epsilon) - y^3\\ \dot{\epsilon} &= 0.\end{aligned}$$

The function $h(y,\epsilon)$ must now be determined. It is clear that $(0,0,\epsilon)$ is a critical point for any $\epsilon \in \mathbb{R}$; hence, $h(0,\epsilon) \equiv 0$, so by Proposition 2.11 one can write $h(y,\epsilon) = yh_1(y,\epsilon)$. As a consequence of the smoothness of the vector field the function h_1 has a Taylor expansion, which by equation (2.1) is given by

$$h_1(y,\epsilon) = ay + b\epsilon + \mathcal{O}(y^2+\epsilon^2).$$

Since W^{c} is invariant one has that

$$\dot{x} = \frac{\partial h}{\partial y}\dot{y} + \frac{\partial h}{\partial \epsilon}\dot{\epsilon},$$

which yields that

$$-h(y,\epsilon) + h(y,\epsilon)^2 - y^2 = (2ay + b\epsilon + \mathcal{O}(y^2+\epsilon^2))(\epsilon y + yh(y,\epsilon) - y^3.$$

Simplifying the above expression gives

$$-(a+1)y^2 - b\epsilon y + \mathcal{O}((|y|+|\epsilon|)^3) = \mathcal{O}((|y|+|\epsilon|)^3),$$

which necessarily implies that

$$a = -1,\quad b = 0.$$

In conclusion,

$$h(y,\epsilon) = y(-y + \mathcal{O}(y^2+\epsilon^2)).$$

As a consequence, the flow on W^{c} is given by

$$\begin{aligned}\dot{y} &= y(\epsilon - 2y^2 + \mathcal{O}((|y|+|\epsilon|)^3))\\ \dot{\epsilon} &= 0.\end{aligned} \tag{2.2}$$

Set

$$m(y,\epsilon) := \epsilon - 2y^2 + \mathcal{O}((|y|+|\epsilon|)^3).$$

Since $m(0,0) = 0$ and $m_\epsilon(0,0) = 1$, by the Implicit Function Theorem there exists an $\epsilon = \epsilon(y)$ and $y_0 > 0$ such that $m_\epsilon(y,\epsilon(y)) \equiv 0$ for all $|y| < y_0$. By inspection one has that $\epsilon(y) = 2y^2 + \mathcal{O}(|y|^3)$. A depiction of the flow on W^{c}, as well as that for the full flow, will be given in class.

The above example yields what is known as a pitchfork bifurcation [**28**, Chapter 20.1e], which has a normal form given by

$$\dot{x} = x(\lambda \pm x^2). \tag{2.3}$$

This normal form can be achieved from equation (2.2) by properly rescaling t and y, and dropping the higher-order terms. The removal of the higher-order terms is justified via the Implicit Function Theorem. Another typical bifurcation associated with one-dimensional center manifolds is the saddle-node bifurcation [**28**, Chapter 20.1c], which has a normal form given by

$$\dot{x} = \lambda \pm x^2. \tag{2.4}$$

The last typical bifurcation associated with one-dimensional center manifolds is the transcritical bifurcation [**28**, Chapter 20.1d], which has a normal form given by

$$\dot{x} = x(\lambda \pm x). \tag{2.5}$$

2.3.2. Hyperbolic conservation laws

A viscous conservation law is given by

$$u_t + f(u)_x = u_{xx}, \tag{2.6}$$

where $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ is C^∞. A thorough discussion of conservation laws and their importance in applications can be found in [**27**, Part III]. The goal here is to find travelling waves, which are solutions $u(z)$, $z := x - st$, of equation (2.6) which satisfy the asymptotics

$$u(z) \to \begin{cases} u_{\mathrm{L}}, & z \to -\infty \\ u_{\mathrm{R}}, & z \to +\infty. \end{cases} \tag{2.7}$$

In the travelling frame equation (2.6) is written as

$$u_t - su_z + f(u)_z = u_{zz}, \tag{2.8}$$

and the travelling wave will now be a steady-state solution, i.e., a solution to

$$-su_z + f(u)_z = u_{zz}. \tag{2.9}$$

It will be realized as a heteroclinic orbit. The following hypothesis will be required.

Hypothesis 2.13. The function f satisfies:

(a) for all $u \in \mathbb{R}^n$, $\mathrm{D}f(u)$ has distinct real eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u)$$

(the system is strictly hyperbolic)

(b) $\langle \nabla\lambda_j(u), r_j(u)\rangle < 0$ for all $u \in \mathbb{R}^n$, where $r_j(u)$ is the eigenvector associated with the eigenvalue $\lambda_j(u)$ (the system is genuinely nonlinear).

Remark 2.14. If $n = 1$, the second condition is equivalent to specifying that $f(u)$ is convex.

Lemma 2.15. *For each $u_{\mathrm{L}} \in \mathbb{R}^n$ and each $1 \le k \le n$ there exists a curve of states $u_{\mathrm{R}}^k(\rho)$ for $0 < \rho < \rho_0$ such that a travelling wave exists with speed $s = s^k(\rho)$. Furthermore,*

(a) $u_{\mathrm{R}}^k(\rho)$ and $s^k(\rho)$ are C^r for any $r \in \mathbb{N}$, and

$$\lim_{\rho\to0^+} u_{\mathrm{R}}^k(\rho) = u_{\mathrm{L}}, \quad \lim_{\rho\to0^+} s^k(\rho) = \lambda_k(u_{\mathrm{L}})$$

(b) $\lambda_k(u_{\mathrm{R}}^k(\rho)) < s^k(\rho) < \lambda_k(u_{\mathrm{L}})$

(c) $\lambda_{k-1}(u_{\mathrm{L}}) < s^k(\rho) < \lambda_{k+1}(u_{\mathrm{R}}^k(\rho))$.

Remark 2.16. Conditions (b) and (c) are known as the Lax entropy inequalities.

Proof: Set $\dot{}\; := \mathrm{d}/\mathrm{d}z$. Integrating equation (2.9) from $-\infty$ to z and using the fact that $u(z) \to u_{\mathrm{L}}$ as $z \to -\infty$ yields

$$\dot{u} = f(u) - f(u_{\mathrm{L}}) - s(u - u_{\mathrm{L}}).$$

Linearizing at the critical point u_{L} yields $A := \mathrm{D}f(u_{\mathrm{L}}) - s\mathbb{1}$. Upon noting that

$$\sigma(A) = \{\lambda - s : \lambda \in \sigma(\mathrm{D}f(u_{\mathrm{L}}))\},$$

it is seen that a bifurcation can occur only if $s = \lambda_k(u_{\mathrm{L}})$ for some $k = 1, \ldots, n$. A branch of solutions will be obtained for each k.

Upon appending $\dot{s} = 0$ to equation (2.9) and linearizing at the point (u_L, s_0), where $s_0 := \lambda_k(u_\mathrm{L})$, one finds that

$$E^\mathrm{c} = \mathrm{span}\{(r_k(u_\mathrm{L}), 0)^\mathrm{T}, (0,1)^\mathrm{T}\}.$$

The equations on W^c must now be computed. The graph of W^c is given by

$$u = u_\mathrm{L} + \eta r_k(u_\mathrm{L}) + W(\eta, s), \tag{2.10}$$

where $W(\eta, s)$ is the complementary direction, i.e.,

$$W(\eta, s) = \sum_{j \neq k} a_j(\eta, s) r_j(u_\mathrm{L}).$$

As a consequence of the center manifold theorem one has that

$$a_j(0, s_0) = \mathrm{D}_s a_j(0, s_0) = 0, \quad \mathrm{D}_\eta a_j(0, s_0) = 0.$$

Let $\ell_j(u)$ be the eigenvectors of $\mathrm{D}f(u)^\mathrm{T}$ which satisfy

$$\langle \ell_i(u), r_j(u) \rangle = \delta_{ij}.$$

Upon taking a Taylor expansion for $f(u)$ at $u = u_\mathrm{L}$ and applying $\langle \ell_k(u_\mathrm{L}), \cdot \rangle$ one sees that

$$\begin{aligned}
\langle \ell_k(u_\mathrm{L}), (u \dot{-} u_\mathrm{L}) \rangle &= \langle \ell_k(u_\mathrm{L}), \mathrm{D}f(u_\mathrm{L})(u - u_\mathrm{L}) \rangle - \langle \ell_k(u_\mathrm{L}), s(u - u_\mathrm{L}) \rangle \\
&\quad + \langle \ell_k(u_\mathrm{L}), \frac{1}{2}\mathrm{D}^2 f(u_\mathrm{L})(u - u_\mathrm{L})^2 \rangle + \cdots .
\end{aligned}$$

As a consequence equation (2.10) and the fact that $\langle \ell_k(u_\mathrm{L}), W(\eta, s) \rangle = 0$, and since

$$(\mathrm{D}f(u_\mathrm{L}) - s\mathbb{1})(u - u_\mathrm{L}) = \eta(s_0 - s) r_k(u_\mathrm{L}) + \sum_{j \neq k} a_j(\eta, s)(\lambda_j(u_\mathrm{L}) - s) r_j(u_\mathrm{L}),$$

one sees that the flow on W^c is given by

$$\begin{aligned}
\dot{\eta} &= (s_0 - s)\eta + \langle \ell_k(u_\mathrm{L}), \frac{1}{2}\mathrm{D}^2 f(u_\mathrm{L}) r_k(u_\mathrm{L})^2 \rangle \eta^2 + \mathcal{O}(|s|^i |\eta|^j), \quad (i + j \geq 3) \\
\dot{s} &= 0.
\end{aligned}$$

The claim is that

$$\ell_k^\mathrm{T}(u_\mathrm{L}) \mathrm{D}^2 f(u_\mathrm{L}) r_k(u_\mathrm{L}) = \nabla \lambda_k(u_\mathrm{L}).$$

To prove this, first note that

$$\ell_k(u)^\mathrm{T} \mathrm{D}f(u) r_k(u) = \lambda_k(u), \quad u \in \mathbb{R}^n.$$

Upon differentiating with respect to u, evaluating at $u = u_\mathrm{L}$, and noting that

$$\begin{aligned}
\mathrm{D}\ell_k^\mathrm{T}(u_\mathrm{L}) \mathrm{D}f(u_\mathrm{L}) r_k(u_\mathrm{L}) + \ell_k^\mathrm{T}(u_\mathrm{L}) \mathrm{D}f(u_\mathrm{L}) \mathrm{D}r_k(u_\mathrm{L}) &= \lambda_k(u_\mathrm{L})(\mathrm{D}\ell_k^\mathrm{T}(u_\mathrm{L}) r_k(u_\mathrm{L}) + \ell_k^\mathrm{T}(u_\mathrm{L}) \mathrm{D}r_k(u_\mathrm{L})) \\
&= \lambda_k(u_\mathrm{L}) \mathrm{D}_u \langle \ell_k(u), r_k(u) \rangle |_{u = u_\mathrm{L}} \\
&= 0,
\end{aligned}$$

yields the desired result. The flow on W^c can now be written as

$$\begin{aligned}
\dot{\eta} &= (s_0 - s)\eta + \frac{1}{2}\langle \nabla \lambda_k(u_\mathrm{L}), r_k(u_\mathrm{L}) \rangle \eta^2 + \mathcal{O}(|s|^i |\eta|^j), \quad (i + j \geq 3) \\
\dot{s} &= 0.
\end{aligned}$$

Since the system is genuinely nonlinear, it is not necessary to calculate the terms of $\mathcal{O}(|s|^i |\eta|^j)$ for $i+j \geq 3$. Thus, the bifurcation is of transcritical type. As an application of the Implicit Function Theorem the critical points on W^c are given by $\eta = 0\,(u = u_\mathrm{L})$ and

$$s = s_0 + \frac{1}{2}\langle \nabla \lambda_k(u_\mathrm{L}), r_k(u_\mathrm{L}) \rangle \eta + \mathcal{O}(\eta^2).$$

Let $\rho_0 > 0$ be sufficiently small, and assume that $|s - s_0| < \rho_0$. Upon parameterizing the above curve, one has that for each $|\rho| < \rho_0$ there exists an $\eta_\mathrm{R} = \eta_\mathrm{R}(\rho)$ and $s = s(\rho)$ such that on W^c, $\eta = 0$ is connected to η_R at $s = s(\rho)$.

Now, in order that the solution approach $\eta = 0$ as $z \to -\infty$, one must necessarily have that $s < s_0$. By construction, one has that

$$u_\mathrm{R} = u_\mathrm{L} + \eta_\mathrm{R} r_k(u_\mathrm{L}) + W(\eta_\mathrm{R}, s).$$

Upon performing a Taylor expansion for $\lambda_k(u)$ at $u = u_\mathrm{L}$ and using the above expansion for u_R one sees that

$$\lambda_k(u_\mathrm{R}) = \lambda_k(u_\mathrm{L}) + \langle \nabla\lambda_k(u_\mathrm{L}), r_k(u_\mathrm{L})\rangle \eta_\mathrm{R} + \mathcal{O}(\eta_\mathrm{R}^2).$$

As a consequence, one has that

$$s - \lambda_k(u_\mathrm{R}) = -\frac{1}{2}\langle \nabla\lambda_k(u_\mathrm{L}), r_k(u_\mathrm{L})\rangle \eta_\mathrm{R} + \mathcal{O}(\eta_\mathrm{R}^2),$$

which, since the system is genuinely nonlinear, implies that $s > \lambda_k(u_\mathrm{R})$.

The proof that the second Lax entropy condition follows from the strict hyperbolicity of the system will be left to the interested student. □

2.3.3. Bistable reaction-diffusion equation

Before we go to the example, we first need a preliminary result. For the system

$$\dot{x} = f(x), \quad x(0) = x_0,$$

recall that the flow is denoted by $\phi_t(x_0)$. Let $y(s)$ be a C^1 curve such that $y(0) = x_0$, and let $v := \mathrm{d}y/\mathrm{d}s(0)$. By the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}s}\phi_t(y(s))|_{s=0} = \mathrm{D}_x\phi_t(x_0)v,$$

so that $\mathrm{D}_x\phi_t(x_0)$ takes tangent vectors to curves of initial conditions to tangent vectors of the image. Now, upon using the smoothness of the flow one has that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}_x\phi_t(x_0)v = \mathrm{D}_x\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(x_0)v = \mathrm{D}_x f(\phi_t(x_0))v = \mathrm{D}f(\phi_t(x_0))\mathrm{D}_x\phi_t(x_0)v;$$

hence, $\mathrm{D}_x\phi_t(x_0)v$ solves the equations of variation

$$\dot{\xi} = \mathrm{D}f(\phi_t(x_0))\xi.$$

The utility of this result is that one now has a way of following the flow of vectors tangent to the stable and unstable manifolds.

Now, the techniques and ideas covered in this example are not presented in [**28**]. The interested student should consult [**12**–**17**, **20**] and the references therein for a small subset of other examples. Consider

$$u_t = u_{xx} + f(u), \tag{2.11}$$

where

$$f(u) = u(u-a)(1-u), \quad 0 < a < \frac{1}{2}.$$

It can be shown that $u \equiv 0$ and $u \equiv 1$ are attractors for equation (2.11), whereas $u \equiv a$ is unstable. In the context of mathematical biology, in which equation (2.11) is known as Nagumo's equation, the solution $u \equiv 0$ represents the rest state of a neuron, the solution $u \equiv 1$ represents the active state of a neuron, and $u(x,t)$ is the voltage across the membrane. The goal will be to find travelling waves $u(z)$, $z := x - ct$, which are transitions between the two attractors. The wave can be thought of as approximating the "front" of the nerve impulse. The interested student should consult [**3**–**6**, **11**] and the references therein, or the web site of James Murray, for more details on the biological application.

In travelling coordinates the wave must satisfy

$$\ddot{u} + c\dot{u} + f(u) = 0, \quad \dot{} := \frac{\mathrm{d}}{\mathrm{d}z}, \tag{2.12}$$

which as a system can be written as

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= -cv - f(u). \end{aligned} \tag{2.13}$$

Note that if a wave exists, then the wave speed must satisfy

$$c\int_{-\infty}^{+\infty} \dot{u}(z)^2 \,\mathrm{d}z = -\int_0^1 f(u)\,\mathrm{d}u,$$

which, since $0 < a < 1/2$, implies that $c < 0$.

Lemma 2.17. *There exists a unique $c^* < 0$ for which equation (2.13) has a solution which satisfies*

$$u(z) \to \begin{cases} 0, & z \to -\infty \\ 1, & z \to +\infty. \end{cases}$$

Furthermore, the solution is unique up to spatial translation.

Proof: The critical points of equation (2.13) are $(0,0)$, $(a,0)$, and $(1,0)$. The desired travelling wave satisfies

$$(u(z), v(z)) \to \begin{cases} (0,0), & z \to -\infty \\ (1,0), & z \to +\infty. \end{cases}$$

Linearizing at $(0,0)$ gives the eigenvalues

$$\lambda_0^{\pm} := \frac{1}{2}(-c \pm \sqrt{c^2 - 4f'(0)}),$$

and linearizing at $(1,0)$ gives the eigenvalues

$$\lambda_1^{\pm} := \frac{1}{2}(-c \pm \sqrt{c^2 - 4f'(1)}).$$

Since $f'(0), f'(1) < 0$ one has that both $\lambda_{0,1}^- < 0 < \lambda_{0,1}^+$. Thus, for a fixed value of c there exists a one-dimensional $W_{0,1}^{\mathrm{u}}$ and $W_{0,1}^{\mathrm{s}}$ at $(0,0)$ and $(1,0)$, and these manifolds are tangent to $E_{0,1}^{\mathrm{s}}$ and $E_{0,1}^{\mathrm{u}}$, where

$$E_{0,1}^{\mathrm{s}} = \mathrm{span}\{(1, \lambda_{0,1}^-)^{\mathrm{T}}\}, \quad E_{0,1}^{\mathrm{u}} = \mathrm{span}\{(1, \lambda_{0,1}^+)^{\mathrm{T}}\}.$$

Note that the desired travelling wave must satisfy $W_0^{\mathrm{u}}(c^*) \cap W_1^{\mathrm{s}}(c^*) \neq \varnothing$ for some $c^* < 0$.

Set

$$\mathcal{L} := \{(u,v) : u = a,\, v \geq 0\}, \quad \mathcal{K} := \{(u,v) : 0 \leq u \leq a,\, v \geq 0\}.$$

One has that $(1, \lambda_0^+)^{\mathrm{T}}$ points into $\mathcal{K}$; thus, as a consequence of the unstable manifold theorem, $W_0^{\mathrm{u}}(c)$ enters $\mathcal{K}$. An examination of the vector field on $\partial\mathcal{K}$ yields that $W_0^{\mathrm{u}}(c)$ can leave $\mathcal{K}$ only by crossing $\mathcal{L}$ or by having $v \to +\infty$. Since $\dot{u} = v$, the second scenario is precluded. Furthermore, since $\dot{u} = v$ on $\mathcal{L}$, one can have that $W_0^{\mathrm{u}}(c)$ intersects $\mathcal{L}$ only once. Finally, $W_0^{\mathrm{u}}(c) \cap \mathcal{L} \neq \varnothing$, for otherwise an application of the Poincare-Bendixson theorem yields a closed orbit in $\mathcal{K}$, which is impossible because $\mathrm{int}(\mathcal{K})$ contains no critical points. Similarly, $W_1^{\mathrm{s}}(c)$ intersects $\mathcal{L}$, and does so only one time.

As a consequence of the fact that λ_0^- and λ_1^+ are smooth in c, it can be shown that $W_0^{\mathrm{u}}(c)$ and $W_1^{\mathrm{s}}(c)$ are smooth. Set

$$g^{\mathrm{u}}(c) := W_0^{\mathrm{u}}(c) \cap \mathcal{L}, \quad g^{\mathrm{s}}(c) := W_1^{\mathrm{s}}(c) \cap \mathcal{L}.$$

These functions are well-defined and smooth; furthermore, a travelling wave exists whenever $g^{\mathrm{u}}(c) = g^{\mathrm{s}}(c)$. First suppose that $c = 0$. The system then has a first integral given by

$$E(u, v) := \frac{1}{2}v^2 + \int_0^u f(s)\,\mathrm{d}s.$$

Since $a < 1/2$, one sees that $g^{\mathrm{u}}(0) < g^{\mathrm{s}}(0)$. Now fix $\alpha \in \mathbb{R}^+$, and set

$$\mathcal{K}_\alpha := \{(u, v) : 0 \le u \le 1,\, v \ge \alpha u\}.$$

There exists an $M \in \mathbb{R}^+$ such that $|f(u)/u| < M$ for $u \in [0, 1]$; thus, on the line $v = \alpha u$ one has that

$$\frac{\dot{v}}{\dot{u}} = -c - \frac{f(u)}{\alpha u} > -c - \frac{M}{\alpha}.$$

Choose $\tilde{c}_1 < 0$ so that if $c < \tilde{c}_1$, then $\dot{v}/\dot{u} > \alpha$. For these values of c one then has that the vector field points into $\mathcal{K}_\alpha$ on the line $v = \alpha u$. Now choose $\tilde{c}_2$ so that $\lambda_0^+(c) > \alpha$ for $c < \tilde{c}_2$. If $c < \min\{\tilde{c}_1, \tilde{c}_2\}$, then $W_0^{\mathrm{u}}(c) \subset \mathcal{K}_\alpha$, and hence intersects $\mathcal{L}$ inside of $\mathcal{K}_\alpha$. Now, $W_1^{\mathrm{s}}(c)$ cannot enter $\mathcal{K}_\alpha$, for otherwise the trajectory on $W_1^{\mathrm{s}}(c)$ could not go to $(1, 0)$ as $z \to +\infty$. Thus, $g^{\mathrm{s}}(c) < g^{\mathrm{u}}(c)$ for $c < \min\{\tilde{c}_1, \tilde{c}_2\}$. By the Intermediate Value Theorem there then exists a $c^* < 0$ such that $g^{\mathrm{s}}(c^*) = g^{\mathrm{u}}(c^*)$.

It must now be shown that c^* is unique. First, it is an exercise to show that $g^{\mathrm{u}}(c) > g^{\mathrm{s}}(c)$ for $c \in \mathbb{R}^+$. The uniqueness for $c \in \mathbb{R}^-$ will follow if it can be shown that $g^{\mathrm{u}}(c)$ and $g^{\mathrm{s}}(c)$ are monotone. Append $\dot{c} = 0$ to equation (2.13), let $\xi := (\delta u, \delta v, \delta c)^{\mathrm{T}}$, and recall that the equations of variation,

$$\begin{aligned}
\dot{\delta u} &= \delta v \\
\dot{\delta v} &= -f'(u)\,\delta u - c\,\delta v - v\,\delta c \\
\dot{\delta c} &= 0,
\end{aligned}$$

carry tangent vectors to tangent vectors. One solution to the variational equations is $\xi_1 := (\dot{u}, \dot{v}, 0)^{\mathrm{T}}$, where $(u(z), v(z), c)^{\mathrm{T}}$ is any solution to the ODE. Let ξ_2 be any other solution to the variational equations such that $\{\xi_1, \xi_2\}$ is a linearly independent set. We will later set ξ_2 to be tangent to either

$$W_0^{\mathrm{cu}} := \bigcup_{c \in \mathbb{R}^-} W_0^{\mathrm{u}}(c) \qquad \text{or} \qquad W_0^{\mathrm{cs}} := \bigcup_{c \in \mathbb{R}^-} W_1^{\mathrm{s}}(c).$$

The vector $\xi_1 \times \xi_2$ is perpendicular to $\mathrm{span}\{\xi_1, \xi_2\}$; hence, we will compute an ODE for this vector. In particular, we will compute the ODE for $w := (\xi_1 \times \xi_2) \cdot \mathbf{e}_3$. Since $\dot{\delta c} = 0$, without loss of generality one can set $\xi_2 = (y_1, y_2, 1)^{\mathrm{T}}$. This then implies that $w = \dot{u}y_2 - \dot{v}y_1$, which upon using the variational equations gives

$$\begin{aligned}
\dot{w} &= \ddot{u}y_2 - \ddot{v}y_1 + \dot{u}\dot{y}_2 - \dot{v}\dot{y}_1 \\
&= -cw - v^2
\end{aligned}$$

(recall that $\dot{u} = v$).

Suppose that

$$\xi_2(0) = (0, \frac{\mathrm{d}g^{\mathrm{u}}}{\mathrm{d}c}, 1)^{\mathrm{T}},$$

i.e., $\xi_2(0)$ is tangent to $g^{\mathrm{u}}(c)$, and hence W_0^{cu}. Since ξ_1 and ξ_2 are tangent to W_0^{cu}, there exists a $C \in \mathbb{R}\backslash\{0\}$ such that

$$\lim_{z \to -\infty} (\xi_1 \times \xi_2)\mathrm{e}^{-\lambda_0^+ z} = C(\lambda_0^+, -1, 0)^{\mathrm{T}};$$

in particular,

$$\lim_{z \to -\infty} w\mathrm{e}^{-\lambda_0^+ z} = 0.$$

Since $v(z)\mathrm{e}^{-\lambda_0^+ z} = \mathcal{O}(1)$ as $z \to -\infty$, this yields that

$$w(z) = -\mathrm{e}^{-cz} \int_{-\infty}^{z} \mathrm{e}^{cs} v^2(s)\,\mathrm{d}s.$$

Since

$$w(0) = v(0)\frac{\mathrm{d}g^{\mathrm{u}}}{\mathrm{d}c},$$

one finally sees that

$$\frac{\mathrm{d}g^{\mathrm{u}}}{\mathrm{d}c} = -\frac{1}{v(0)}\int_{-\infty}^{0} \mathrm{e}^{cs}v^2(s)\,\mathrm{d}s < 0.$$

In a similar manner one finds that

$$\frac{\mathrm{d}g^{\mathrm{s}}}{\mathrm{d}c} = \frac{1}{v(0)}\int_{0}^{+\infty} \mathrm{e}^{cs}v^2(s)\,\mathrm{d}s > 0.$$

Thus, both curves are monotone, so that $c^* < 0$ is unique. □

Remark 2.18. Note that when $c = c^*$, $g^{\mathrm{u}} = g^{\mathrm{s}}$ with

$$\frac{\mathrm{d}}{\mathrm{d}c}(g^{\mathrm{u}} - g^{\mathrm{s}}) = -\frac{1}{v(0)}\int_{-\infty}^{+\infty} \mathrm{e}^{cs}v^2(s)\,\mathrm{d}s.$$

When discussing Melnikov theory, it will be important to understand and evaluate quantities such as that given above.

Corollary 2.19. *Consider*

$$\ddot{u} + c\dot{u} + f(u) + \epsilon g(u, \dot{u}, \epsilon) = 0,$$

where $g : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}^+$ is smooth. There exists an $\epsilon_0 > 0$ such that if $0 \le \epsilon < \epsilon_0$, then there exists a unique travelling wave with speed $c(\epsilon)$, where $c(\epsilon)$ is smooth with $c(0) = c^*$.

Proof: Construct the curves $g^{\mathrm{u}}(c,\epsilon)$ and $g^{\mathrm{s}}(c,\epsilon)$ as in the proof of Lemma 2.17. It has already been seen that

$$g^{\mathrm{u}}(c^*, 0) - g^{\mathrm{s}}(c^*, 0) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}c}(g^{\mathrm{u}}(c^*, 0) - g^{\mathrm{s}}(c^*, 0)) < 0.$$

By the Implicit Function Theorem there exists an $\epsilon_0 > 0$ such that for $0 <\le \epsilon < \epsilon_0$ there is a smooth $c(\epsilon)$ with $c(0) = c^*$ such that

$$g^{\mathrm{u}}(c(\epsilon), \epsilon) - g^{\mathrm{s}}(c(\epsilon), \epsilon) = 0.$$ □

3. Melnikov's Method

In this section we will consider another technique to show the persistence of heteroclinic and homoclinic orbits. As in the example of Section 2.3.3, the result will depend heavily upon the Implicit Function Theorem. The material presented in this section is also covered in [**28**, Chapter 28].

3.1. Preliminary result

Consider the system

$$\dot{y} = A(t)y + g(t), \tag{3.1}$$

where

(a) $A(t) \in \mathbb{R}^{2\times 2}$, $g(t) \in \mathbb{R}^2$ are continuous, and $g(t)$ is uniformly bounded

(b) $\lim_{|t|\to\infty} A(t) = A_0$, with the approach being exponentially fast

(c) trace $A(t) \equiv 0$ (not an essential assumption)

(d) $A_0 = P\,\mathrm{diag}(\lambda, -\lambda)P^{-1}$ where $\lambda \in \mathbb{R}^+$.

Now assume that there exists a solution $y_{\mathrm{h}}(t)$ to the homogeneous system

$$\dot{y} = A(t)y \tag{3.2}$$

such that $|y_{\mathrm{h}}(t)|\mathrm{e}^{\lambda t} \le C$, i.e., $y_{\mathrm{h}}(t)$ decays exponentially fast as $|t| \to \infty$. Define

$$d(v_1, \dots, v_n) := \det(v_1, \dots, v_n), \quad v_1, \dots, v_n \in \mathbb{R}^n,$$

and let $y_2(t)$ be the solution to equation (3.2) which satisfies $d(y_{\mathrm{h}}(0), y_2(0)) = 1$. As a consequence of Abel's theorem one has that $d(y_{\mathrm{h}}(t), y_2(t)) \equiv 1$, which implies that $|y_2(t)|\mathrm{e}^{-\lambda t} \ge C$, i.e., $y_2(t)$ grows exponentially fast as $|t| \to \infty$.

We shall now construct a special set of solutions, $y^{\pm}(t)$, to equation (3.1). In particular, we wish to have $y^{+}(t)\,(y^{-}(t))$ be bounded for $t \ge 0\,(t \le 0)$. Upon applying variation of parameters to equation (3.1) one can check that the desired solutions are given by

$$y^{\pm}(t) = c^{\pm} y_{\mathrm{h}}(t) + y_{\mathrm{h}}(t) \int_0^t d(g(s), y_2(s))\,\mathrm{d}s + y_2(t) \int_{\pm\infty}^t d(y_{\mathrm{h}}(s), g(s))\,\mathrm{d}s, \tag{3.3}$$

where $c^{\pm} \in \mathbb{R}$. It can be checked that the assumption on $y_{\mathrm{h}}(t)$ guarantees that these solutions are appropriately bounded; furthermore, if $|g(t)| \to 0$ sufficiently fast as $|t| \to \infty$, then $|y^{\pm}(t)| \to 0$ as $t \to \pm\infty$.

As an observation, note that

$$(y^{-} - y^{+})(t_0) = (c^{-} - c^{+}) y_{\mathrm{h}}(t_0) + y_2(t_0) \int_{-\infty}^{+\infty} d(y_{\mathrm{h}}(s), g(s))\,\mathrm{d}s.$$

Now consider the adjoint equation to equation (3.2),

$$\dot{z} = -A^{\mathrm{T}}(t) z. \tag{3.4}$$

Let $z_{\mathrm{h}}(t)$ and $z_2(t)$ be two solutions which satisfy

$$\begin{aligned} \langle z_2(0), y_{\mathrm{h}}(0)\rangle &= 1, \quad \langle z_2(0), y_2(0)\rangle = 0 \\ \langle z_{\mathrm{h}}(0), y_{\mathrm{h}}(0)\rangle &= 0, \quad \langle z_{\mathrm{h}}(0), y_2(0)\rangle = 1. \end{aligned}$$

Since $\langle z(t), y(t)\rangle = C$ for any solutions $y(t)$ to equation (3.2) and $z(t)$ to equation (3.4), one has that

$$|z_2(t)|\mathrm{e}^{-\lambda|t|} \ge C, \quad |z_{\mathrm{h}}(t)|\mathrm{e}^{\lambda|t|} \le C.$$

Furthermore, one can rewrite the result of equation (3.3) to say that

$$\langle z_{\mathrm{h}}(t_0), (y^{-} - y^{+})(t_0)\rangle = \int_{-\infty}^{+\infty} d(y_{\mathrm{h}}(s), g(s))\,\mathrm{d}s. \tag{3.5}$$

3.2. Homoclinic bifurcations

Consider the system

$$\dot{x} = f(x) + \epsilon g(x, \epsilon),$$

where $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ and $g : \mathbb{R}^2 \times \mathbb{R} \mapsto \mathbb{R}^2$ are smooth. Suppose that

(a) $f(0) = 0$ and $x = 0$ is a saddle point

(b) $\nabla \cdot f(x) \equiv 0$

(c) when $\epsilon = 0$ there exists an orbit $q_0(t)$ homoclinic to $x = 0$.

The goal of this subsection is to consider the persistence of $q_0(t)$ under the perturbation. Before doing so, we need the following preliminary result.

Lemma 3.1. *There is an $\epsilon_0 > 0$ such that for all $|\epsilon| < \epsilon_0$ there exists an x_ϵ with $x_0 = 0$ such that*

$$f(x_\epsilon) + \epsilon g(x_\epsilon, \epsilon) = 0.$$

Furthermore, for each $|\epsilon| < \epsilon_0$ there exists one-dimensional manifolds $W^{\mathrm{u}}(x_\epsilon)$ and $W^{\mathrm{s}}(\epsilon)$, and these manifolds are smooth in ϵ.

Proof: The persistence of the critical point follows from setting

$$h(x, \epsilon) := f(x) + \epsilon g(x, \epsilon),$$

noting that $h(0, 0) = 0$ and $\mathrm{D}_x h(0, 0) = \mathrm{D}f(0)$ is invertible, and invoking the Implicit Function Theorem. The existence and smoothness of the manifolds follows from the manifold theorems. □

In order for the homoclinic orbit to persist, one needs that $W^{\mathrm{u}}(x_\epsilon) \cap W^{\mathrm{s}}(\epsilon) \neq \varnothing$. We now proceed to make the calculation that will ensure this intersection. Let $x_\epsilon^{\mathrm{u}}(t)$ $(x_\epsilon^{\mathrm{s}}(t))$ represent the trajectory on $W^{\mathrm{u}}(x_\epsilon)$, $(W^{\mathrm{s}}(\epsilon))$, and recall that by assumption, $x_0^{\mathrm{u}}(t) = x_0^{\mathrm{s}}(t) = q_0(t)$. As a consequence of smoothness one can write

$$\begin{aligned} x_\epsilon^{\mathrm{u}}(t) &= q_0(t) + \epsilon x_1^{\mathrm{u}}(t) + \mathcal{O}(\epsilon^2), \quad t \le 0 \\ x_\epsilon^{\mathrm{s}}(t) &= q_0(t) + \epsilon x_1^{\mathrm{s}}(t) + \mathcal{O}(\epsilon^2), \quad t \ge 0, \end{aligned}$$

where the functions $x_1^{\mathrm{u,s}}$ are uniformly bounded on their domains. Upon noting that

$$(x_\epsilon^{\mathrm{s}} - x_\epsilon^{\mathrm{u}})(0) = \epsilon(x_1^{\mathrm{s}} - x_1^{\mathrm{u}})(0) + \mathcal{O}(\epsilon^2), \tag{3.6}$$

one sees that a necessary condition for persistence is that $(x_1^{\mathrm{s}} - x_1^{\mathrm{u}})(0) = 0$.

Since the manifolds are invariant, one has that

$$\dot{x}_\epsilon^{\mathrm{s,u}} = f(x_\epsilon^{\mathrm{s,u}}) + \epsilon g(x_\epsilon^{\mathrm{s,u}}, \epsilon);$$

thus, upon differentiating with respect to ϵ and evaluating at $\epsilon = 0$ one sees that

$$\begin{aligned} \dot{x}_1^{\mathrm{u}} &= \mathrm{D}f(q_0) + g(q_0, 0), \quad t \le 0 \\ \dot{x}_1^{\mathrm{s}} &= \mathrm{D}f(q_0) + g(q_0, 0), \quad t \ge 0. \end{aligned}$$

These equations must be solved so that $x_1^{\mathrm{s,u}}$ are uniformly bounded on their respective domains. Since one solution to the homogeneous equation $\dot{y} = \mathrm{D}f(q_0)y$ is $y_{\mathrm{h}} = \dot{q}_0$ $(= f(q_0))$. As a consequence of the discussion in Section 3.1 one has that

$$(x_1^{\mathrm{s}} - x_1^{\mathrm{u}})(0) = c\dot{q}_0(0) + M y_2(0), \tag{3.7}$$

where the Melnikov function M is given by

$$M := \int_{-\infty}^{+\infty} d(f(q_0(s)), g(q_0(s), 0)\, \mathrm{d}s. \tag{3.8}$$

Now, as in Section 3.1 let $z_{\mathrm{h}}(t)$ be the solution to the adjoint equation $\dot{z} = -\mathrm{D}f(q_0)^{\mathrm{T}} z$ which satisfies

$$\langle z_{\mathrm{h}}(0), \dot{q}_0(0)) \rangle = 0;$$

in particular, note that $z_{\mathrm{h}}(0)$ is orthogonal to the vector field. As a consequence, if

$$M = \langle z_{\mathrm{h}}(0), (x_1^{\mathrm{s}} - x_1^{\mathrm{u}})(0) \rangle \tag{3.9}$$

is such that $M = 0$, then up to $\mathcal{O}(\epsilon^2)$ one has that $W^{\mathrm{u}}(x_\epsilon) \cap W^{\mathrm{s}}(x_\epsilon) \neq \varnothing$.

Theorem 3.2. *Let $M : \mathbb{R} \mapsto \mathbb{R}$ be defined as in equation (3.8), and suppose that $M(\mu_0) = 0$ with $M'(\mu_0) \neq 0$. There then exists an $\epsilon_0 > 0$ such that for all $|\epsilon| < \epsilon_0$ there exists a unique $\mu(\epsilon)$ with $\mu(0) = \mu_0$ such that $W^{\mathrm{u}}(x_\epsilon) \cap W^{\mathrm{s}}(x_\epsilon) \neq \varnothing$ for $\mu = \mu(\epsilon)$.*

Proof: As a consequence of equation (3.6) and equation (3.9) we have that

$$\langle z_{\mathrm{h}}(0), (x_\epsilon^{\mathrm{s}} - x_\epsilon^{\mathrm{u}})(0)\rangle = \epsilon(M(\mu) + \mathcal{O}(\epsilon)).$$

The result now follows upon applying the Implicit Function Theorem to $g(\mu, \epsilon) := M(\mu) + \mathcal{O}(\epsilon)$. □

Remark 3.3. One has that

(a) if $\nabla \cdot f(x) \neq 0$, then the Melnikov function is given by

$$M = \int_{-\infty}^{+\infty} \mathrm{e}^{-\int_0^s \nabla\cdot f(q_0(r))\,\mathrm{d}r} d(f(q_0(s)), g(q_0(s), 0)\,\mathrm{d}s$$

(b) the theory is also applicable when studying the persistence of heteroclinic orbits

(c) there is an analogous theory for the system

$$\dot{x} = f(x) + \epsilon g(x, t, \epsilon),$$

where $g(x, t+T, \epsilon) = g(x, t, \epsilon)$ for some $T > 0$ (see [**28**, Chapter 28] and [**9**, Chapter 4]).

The last remark is especially pertinent, and the implications of periodicity in the forcing function will be explored more fully at a later date.

3.3. Example: Takens-Bogdanov bifurcation

Consider $\dot{x} = f(x, \mu)$, and assume that $f(0, \mu) \equiv 0$. Furthermore, assume that for $A := \mathrm{D}_x f(0, 0)$ one has that $\sigma^{\mathrm{c}}(A) = \{0\}$, and that this eigenvalue has geometric multiplicity one and algebraic multiplicity two. As discussed in [**28**, Chapter 20.6,Chapter 33.1], the normal form associated with flow on W^{c} is given by

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= \mu_1 + \mu_2 y + x^2 + bxy, \quad b \in \{-1, +1\}.\end{aligned} \tag{3.10}$$

For $\epsilon > 0$ introduce the scalings

$$x := \epsilon^2 u, \quad y := \epsilon^3 v, \quad \mu_1 := -\epsilon^4, \quad \mu_2 := \epsilon^2 \nu_2, \quad t := \epsilon s,$$

so that equation (3.10) becomes

$$\begin{aligned}\dot{u} &= v \\ \dot{v} &= -1 + u^2 + \epsilon(\nu_2 v + buv).\end{aligned} \tag{3.11}$$

When $\epsilon = 0$ equation (3.11) is a completely integrable Hamiltonian system with Hamiltonian

$$H(u, v) := \frac{1}{2}v^2 + u - \frac{1}{3}u^3.$$

The system has two critical points: the point $(1, 0)$ is a saddle point which has the homoclinic orbit $(u_0(t), v_0(t))$, where

$$u_0(t) = 1 - 3\,\mathrm{sech}^2(t/\sqrt{2}),$$

and the point $(-1, 0)$ is a nonlinear center.

The result of Theorem 3.2 can now be used to determine the persistence of the homoclinic orbit. The Melnikov function is given by

$$\begin{aligned}M(\nu_2) &= \nu_2 \int_{-\infty}^{+\infty} v_0^2(t)\,\mathrm{d}t + b\int_{-\infty}^{+\infty} u_0(t) v_0^2(t)\,\mathrm{d}t \\ &= 7\nu_2 - 5b.\end{aligned}$$

Hence, the orbit will persist for $\nu_2 = \nu_2^*$, where

$$\nu_2^* := \frac{5}{7}b + \mathcal{O}(\epsilon).$$

Note that for $\nu_2 < \nu_2^*$, $M < 0$, while for $\nu_2 > \nu_2^*$, $M > 0$. This yields the relative orientation of the stable and unstable manifolds. Further note that with respect to the original variables, the orbit persists for

$$\mu_1 = -\frac{49}{25}\mu_2^2 + \mathcal{O}(\mu_2^{5/2}),$$

and that $x(t) = \mathcal{O}(\mu_2)$.

4. Hopf Bifurcation

In Section 3.3 we considered one type of planar bifurcation; in particular, the existence and persistence of homoclinic orbits. Herein we will now consider the creation of periodic orbits via what is known as a Hopf bifurcation. The technique behind the proof will require knowledge of a particular type of bifurcation associated with maps, as we will construct the bifurcating periodic solution via a Poincaré map.

4.1. Period doubling in maps

Consider the discrete dynamical system

$$x_{n+1} = f(x_n, \epsilon), \tag{4.1}$$

where $f : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$ is a smooth diffeomorphism. Using the notation $f^k := f \circ f \cdots \circ f$ (k times), we say that

$$\gamma^{\pm}(x_0) := \{x_0, f^{\pm 1}(x_0, \epsilon), \dots, f^{\pm k}(x_0, \epsilon), \dots\}$$

are the positive and negative orbits of x_0, and $\gamma(x_0) := \gamma^-(x_0) \cup \gamma^+(x_0)$ is the orbit of x_0. We say that $y \in \omega(x_0)$ if there is a sequence $\{n_i\}$ with $\lim n_i = +\infty$ such that $y = \lim f^{n_i}(x_0, \epsilon)$, and the set $\omega(x_0)$ is the union of all such points. The set $\alpha(x_0)$ is defined by letting $\lim n_i = -\infty$.

Finally, a set $M \subset \mathbb{R}^n$ is invariant if $f(M, \epsilon) = M$, i.e., if for each $x \in M$ one has that $f(x, \epsilon) \in M$, and if for each $y \in M$ there exists an $x \in M$ such that $f(x, \epsilon) = y$. One important example of an invariant set is a period-N orbit $\{x_0, f(x_0, \epsilon), \dots, f^N(x_0, \epsilon)\}$, in which $x_0 = f^{N+1}(x_0, \epsilon)$, but $x_0 \neq f^j(x_0, \epsilon)$ for any $1 \leq j \leq N$. In order to see how such an orbit can arise, consider the linear map

$$x_{n+1} = Ax_n, \quad A := \begin{pmatrix} a & b \\ -b & a \end{pmatrix},$$

where $a^2 + b^2 = 1$. If $x = (x_1, x_2)^{\mathrm{T}}$, then upon writing $x_1 := r\cos\theta$, $x_2 := r\sin\theta$ one sees that

$$r_{n+1} = r_n, \quad \theta_{n+1} = \theta_n - \omega,$$

where $\omega \in [0, 2\pi)$ is defined so that $a + \mathrm{i}b = \mathrm{e}^{\mathrm{i}\omega}$. Thus, the circle $r = r_0$ is invariant, and the mapping on this circle is $\theta_n = \theta_0 - n\omega$. If $2\pi/\omega \in \mathbb{Q}$, then for some $N \in \mathbb{N}$ one has that $N\omega = 0 \pmod{2\pi}$, which implies the existence of an N-periodic orbit. If $2\pi/\omega \in \mathbb{R}\backslash\mathbb{Q}$, then the orbit on the circle is dense, i.e., given a point θ^* there is a sequence $\{n_j\}$ with $\lim n_j = +\infty$ such that $\lim(\theta_0 - n_j\omega) = \theta^* \pmod{2\pi}$.

Now consider equation (4.1) in the scalar case, and assume that $f(0, \epsilon) \equiv 0$ with $f_x(0, 0) = -1$. A Taylor expansion then yields

$$f(x, \epsilon) = -x + \frac{1}{2}ax^2 + b\epsilon x + \frac{1}{6}cx^3 + \cdots.$$

If one tries to find nonzero fixed points of $\mathcal{O}(\epsilon)$, then one is left to solving

$$x = -x + \frac{1}{2}ax^2 + b\epsilon x + \frac{1}{6}cx^3 + \cdots,$$

which clearly has no solutions of $\mathcal{O}(|\epsilon|^{1/2})$. If one now tries to find fixed points of

$$f^2(x,\epsilon) = x - 2b\epsilon x - \frac{1}{2}(\frac{1}{3}c + a^2)x^3 + \cdots,$$

then one is left to solving

$$x = x - 2b\epsilon x - \frac{1}{2}(\frac{1}{3}c + a^2)x^3 + \cdots.$$

If $b \neq 0$, then as an application of the Implicit Function Theorem one has that nontrivial solutions lie on the curve

$$\epsilon = -Sx^2 + \mathcal{O}(x^3), \quad S := \frac{c/3 + a^2}{4b},$$

where S is known as the Schwarzian derivative. Thus, fixed points for $f^2(x,\epsilon)$ arise via a pitchfork bifurcation. If $S > 0$ the bifurcation is subcritical, whereas if $S > 0$ the bifurcation is supercritical.

Let the upper curve be denoted by $x = x^{\mathrm{u}}(\epsilon)$, and let the lower curve be denoted by $x = x^{\ell}(\epsilon)$. One has that

$$x^{\mathrm{u}}(\epsilon) = f^2(x^{\mathrm{u}}(\epsilon),\epsilon), \quad x^{\ell}(\epsilon) = f^2(x^{\ell}(\epsilon),\epsilon),$$

which in turn implies that

$$f(x^{\mathrm{u}}(\epsilon),\epsilon) = f^3(x^{\mathrm{u}}(\epsilon),\epsilon), \quad f(x^{\ell}(\epsilon),\epsilon) = f^3(x^{\ell}(\epsilon),\epsilon).$$

Since $f^3 = f^2 \circ f$, this yields that $f(x^{\mathrm{u}}(\epsilon),\epsilon)$ and $f(x^{\ell}(\epsilon),\epsilon)$ are also fixed points for $f^2(x,\epsilon)$. Since $f(x,\epsilon)$ has no fixed points other than $x = 0$, by uniqueness one must have that

$$x^{\ell}(\epsilon) = f(x^{\mathrm{u}}(\epsilon),\epsilon), \quad x^{\mathrm{u}}(\epsilon) = f(x^{\ell}(\epsilon),\epsilon).$$

Thus, under the conditions $f_x(0,0) = -1$ and $b = f_{\epsilon x}(0,0) \neq 0$ we have shown that a period-2 orbit given by $\{x^{\mathrm{u}}(\epsilon), f(x^{\mathrm{u}}(\epsilon),\epsilon)\}$ arises via a pitchfork bifurcation.

Remark 4.1. An alternative interpretation of the above result is that fixed points for f^2 arise via a pitchfork bifurcation. These points arise if

$$f^2(0,\epsilon) \equiv 0, \quad \frac{\partial^2}{\partial x \partial \epsilon} f^2(0,0) = 1.$$

4.2. The Hopf bifurcation theorem

Consider $\dot{x} = f(x,\mu)$, where $f : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$ is smooth. Let $(x_0, 0)$ be a critical point such that $0 \notin \sigma(\mathrm{D}_x f(x_0,0))$. As a consequence of the Implicit Function Theorem, for $|\mu| < \mu^*$ there exists a unique curve of critical points $(x(\mu),\mu)$ with $x(0) = x_0$. Suppose that $\mathrm{D}_x f(x(\mu),\mu)$ has the simple eigenvalues $\alpha(\mu) \pm \mathrm{i}\beta(\mu)$ which satisfy

$$\alpha(0) = 0, \quad \alpha'(0) \neq 0, \quad \beta(0) > 0. \tag{4.2}$$

Further suppose that $\sigma^{\mathrm{c}}(\mathrm{D}_x f(x_0,0)) = \{\pm \mathrm{i}\beta(0)\}$. As discussed in [**28**, Chapter 20.2], the normal form for the equations on W^{c} can then be written as

$$\begin{aligned}\dot{x} &= \alpha(\mu)x - \beta(\mu)y + (a(\mu)x - b(\mu)y)(x^2+y^2) + \mathcal{O}(|x|^5,|y|^5)\\ \dot{y} &= \beta(\mu)x + \alpha(\mu)y + (b(\mu)x + a(\mu)y)(x^2+y^2) + \mathcal{O}(|x|^5,|y|^5).\end{aligned} \tag{4.3}$$

In polar coordinates equation (4.3) can be written as

$$\begin{aligned}\dot{r} &= \alpha(\mu)r + a(\mu)r^3 + \mathcal{O}(r^5)\\ \dot{\theta} &= \beta(\mu) + b(\mu)r^2 + \mathcal{O}(r^4).\end{aligned} \tag{4.4}$$

Note that a T-periodic solution to equation (4.3) is equivalent to having a solution $(r(t),\theta(t))$ to equation (4.4) which satisfies

$$r(0) = r(T), \quad \theta(0) = 0, \quad \theta(T) = 2\pi.$$

Upon taking a Taylor expansion and neglecting the higher-order terms, one finally gets the equations to be studied:

$$\begin{aligned}\dot r &= \alpha'(0)\mu r + a(0)r^3\\ \dot\theta &= \beta(0) + \beta'(0)\mu + b(0)r^2.\end{aligned} \tag{4.5}$$

Theorem 4.2 (Hopf Bifurcation Theorem)**.** *Consider the system equation (4.3) under the constraint given in equation (4.2). If $a(0) \neq 0$ and if $|\mu|$ is sufficiently small, then there exists a unique periodic solution of $\mathcal{O}(|\mu|^{1/2})$.*

Proof: For the initial condition $r(0) = \xi$, $\theta_0 = 0$, denote the solution to equation (4.5) by $r(t, \xi, \mu)$ and $\theta(t, \xi, \mu)$. Since

$$r(t, 0, \mu) \equiv 0, \quad \theta(2\pi/\beta(\mu), 0, \mu) \equiv 2\pi,$$

upon applying the Implicit Function Theorem one obtains a neighborhood $U_0 \times V_0$ of $(r, \mu) = (0, 0)$, and a smooth function $T : U_0 \times V_0 \mapsto \mathbb{R}$ with $T(0, 0) = 2\pi/\beta(0)$, such that $\theta(T(\xi, \mu), \xi, \mu) \equiv 2\pi$. Define the Poincaré map $\Pi : U_0 \times V_0 \mapsto \mathbb{R}$ by

$$\Pi(\xi, \mu) := r(T(\xi, \mu), \xi, \mu),$$

and recall that fixed points of Π yield periodic orbits for equation (4.3).

The goal is to show that Π has the same properties as the map f^2 described in Section 4.1; in particular, that it undergoes a pitchfork bifurcation at the point $(r, \mu) = (0, 0)$. First note that $P(0, \mu) \equiv 0$, and that

$$\frac{\partial \Pi}{\partial \xi}(0, 0) = \frac{\mathrm{d}T}{\mathrm{d}\xi}(0, 0)\dot r + \frac{\mathrm{d}r}{\mathrm{d}\xi}(0, 0).$$

Since $r = 0$ is invariant, one has that $\dot r = 0$. Furthermore, when $r = 0$ one gets that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}r}{\mathrm{d}\xi} = \alpha'(0)\mu\frac{\mathrm{d}r}{\mathrm{d}\xi}, \quad \frac{\mathrm{d}r}{\mathrm{d}\xi}(0) = 1,$$

which for $\mu = 0$ has the solution $\partial r/\partial\xi(t) = 1$. Hence,

$$\frac{\partial \Pi}{\partial \xi}(0, 0) = 1.$$

A tedious calculation reveals that when $(r, \mu) = (0, 0)$,

$$\frac{\partial}{\partial \mu}\frac{\partial \Pi}{\partial \xi} = \frac{\partial}{\partial \mu}\frac{\partial r}{\partial \xi}.$$

When $r = 0$ one has that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial \mu}\frac{\partial r}{\partial \xi}\right) = \alpha'(0)\frac{\partial r}{\partial \xi},$$

which implies that

$$\frac{\partial}{\partial \mu}\frac{\partial \Pi}{\partial \xi} = 2\pi\frac{\alpha'(0)}{\beta(0)} \neq 0.$$

The Poincaré map then undergoes the desired pitchfork bifurcation, which implies the existence of a fixed point of $\mathcal{O}(\mu^{1/2})$.

The uniqueness of the fixed point follows from the fact that $a(0) \neq 0$ (see [**28**, Chapter 20.2]). The interested student should consult [**28**, equation (20.2.14)] for an explicit expression for $a(0)$. □

Remark 4.3. To leading order the Poincaré map is given by

$$\Pi(\xi, \mu) = \left(1 + \mu 2\pi\frac{\alpha'(0)}{\beta(0)}\right)\xi + a(0)\xi^3.$$

If $a(0)\alpha'(0) < 0$, then the bifurcation is supercritical, whereas if $a(0)\alpha'(0) > 0$, then the bifurcation is subcritical.

Remark 4.4. An alternate proof of Theorem 4.2 is given in [**28**, Theorem 20.2.3].

4.3. Example: Takens-Bogdanov bifurcation

Recall the system given in equation (3.10). Assuming that $\mu_1 < 0$, when consider the critical point $(-\sqrt{-\mu_1}, 0)$ the eigenvalues of the linearization are given by

$$\lambda^{\pm} = \frac{1}{2}\left(\mu_2 - \sqrt{-\mu_1} \pm \sqrt{(\mu_2 - \sqrt{-\mu_1})^2 - 8\sqrt{-\mu_1}}\right).$$

If one writes $\mu_2 = \sqrt{-\mu_1} + \mu_2^\epsilon \epsilon$ for $0 \le \epsilon \ll 1$, i.e.,

$$\mu_1 = -\mu_2^2 + 2\mu_2\mu_2^\epsilon\epsilon + \mathcal{O}(\epsilon^2),$$

then one can rewrite the above as

$$\lambda^{\pm} = \frac{1}{2}\left(\mu_2^\epsilon\epsilon \pm \mathrm{i}\sqrt{2} + \mathcal{O}(\epsilon^2)\right).$$

Thus, upon applying Theorem 4.2 one has that a Hopf bifurcation occurs at $\epsilon = 0$. It can be computed that $a(0) = b/16$ (see [**28**, Chapter 20.6]). Since we are requiring that $\epsilon > 0$, for the bifurcation to occur we must have that $b\mu_2^\epsilon < 0$; hence, it is supercritical. If one assumes that $b = -1$, then the bifurcating solution is stable, whereas if $b = +1$ the bifurcating solution is unstable.

5. Maps

While the study of general maps is interesting in its own right, we will primarily focus upon the properties associated with Poincaré maps. The eventual goal of this section is to rigorously study the dynamics with periodically forced Hamiltonian systems. One concrete example of such a system is the periodically forced and damped pendulum, which was discussed in some detail in Math 512.

5.1. Linear maps

First consider the scalar linear map

$$x_{n+1} = ax_n, \quad a \in \mathbb{C}$$

which has the solution

$$x_n = a^n x_0.$$

Upon writing $a := |a|\mathrm{e}^{\mathrm{i}\omega}$ for some $\omega \in [0, 2\pi)$ one sees that

$$x_n = |a|^n \mathrm{e}^{\mathrm{i}n\omega} x_0.$$

Thus, it is seen that if $|a| < 1$ the solution has the behavior that $x_n \to 0$ as $n \to +\infty$, while if $|a| > 1$ the solution has the behavior $x_n \to 0$ as $n \to -\infty$. If $|a| = 1$, then as was seen in Section 4.1 the solution will either be periodic or its trajectory will densely fill the circle with radius $|x_0|$.

Now consider

$$x_{n+1} = Ax_n, \quad A \in \mathbb{R}^{n \times n}, \tag{5.1}$$

which has the solution $x_n = A^n x_0$. Analogously as in Section 2.1, but with less generality, set

$$\begin{aligned}
\sigma^{\mathrm{s}}(A) &:= \{\lambda \in \sigma(A) : |\lambda| < 1\} \\
\sigma^{\mathrm{c}}(A) &:= \{\lambda \in \sigma(A) : |\lambda| = 1\} \\
\sigma^{\mathrm{u}}(A) &:= \{\lambda \in \sigma(A) : |\lambda| > 1\}.
\end{aligned}$$

Associated with each spectral set there is a subspace $E^{\mathrm{s,c,u}}$ which satisfies the property that

$$AE^{\mathrm{s,c,u}} \subset E^{\mathrm{s,c,u}}.$$

Let $0 < \epsilon \ll 1$. One can easily show that for equation (5.1) that there is a $C, M(\epsilon) \ge 1$ and $\alpha \in (0, 1)$ such that if

(a) $x_0 \in E^{\mathrm{s}}$, then
$$|x_n|\alpha^{-n} \le C|x_0|, \quad n \in \mathbb{N}_0$$

(b) $x_0 \in E^{\mathrm{u}}$, then
$$|x_n|\alpha^{n} \le C|x_0|, \quad -n \in \mathbb{N}_0$$

(c) $x_0 \in E^{\mathrm{c}}$, then
$$M(\epsilon)^{-1}(1-\epsilon)^n|x_0| \le |x_n| \le M(\epsilon)(1+\epsilon)^n|x_0|, \quad n \in \mathbb{Z}.$$

5.2. Invariant manifolds

Let us now consider the mapping
$$x_{n+1} = f(x_n), \tag{5.2}$$
where $f \in C^r(\mathbb{R}^n;\mathbb{R}^n)$ $(r \ge 2)$ is a smooth diffeomorphism. Note that the solution is given by $x_n = f^n(x_0)$. It will be assumed that $f(0) = 0$, so that $x = 0$ is a fixed point of the map.

Definition 5.1. One says that $f^n(x) \to 0$ exponentially fast if there exists an $\alpha \in (0,1)$ and a $C \in \mathbb{R}^+$ such that $|f^n(x)|\alpha^{-n} \le C$ for $n \in \mathbb{N}_0$.

Definition 5.2. Let N be a given small neighborhood of $x = 0$, and let $\epsilon > 0$ be sufficiently small. The stable manifold, W^{s}, is
$$W^{\mathrm{s}} := \{x \in N \,:\, f^n(x) \in N\, \forall\, n \in \mathbb{N}_0 \text{ and } f^n(x) \to 0 \text{ exponentially fast as } n \to +\infty\}.$$
The unstable manifold, W^{u}, is
$$W^{\mathrm{u}} := \{x \in N \,:\, f^{-n}(x) \in N\, \forall\, n \in \mathbb{N}_0 \text{ and } f^{-n}(x) \to 0 \text{ exponentially fast as } t \to -\infty\}.$$
The center manifold, W^{c}, is invariant relative to N, i.e., if $x \in W^{\mathrm{c}}$, then $f^{\pm n}(x) \in N$ for $n \in \mathbb{N}_0$. Furthermore, $W^{\mathrm{c}} \cap W^{\mathrm{s}} = W^{\mathrm{c}} \cap W^{\mathrm{u}} = \{0\}$, and
$$\dim W^{\mathrm{c}} + \dim W^{\mathrm{s}} + \dim W^{\mathrm{u}} = n.$$

Theorem 5.3 (Stable manifold theorem)**.** *There is a neighborhood N of $x = 0$ and a C^{r-1} function $h^{\mathrm{s}} : N \cap E^{\mathrm{s}} \mapsto E^{\mathrm{c}} \oplus E^{\mathrm{u}}$ such that $W^s = \mathrm{graph}(h^{\mathrm{s}})$.*

Theorem 5.4 (Unstable manifold theorem)**.** *There is a neighborhood N of $x = 0$ and a C^{r-1} function $h^{\mathrm{u}} : N \cap E^{\mathrm{u}} \mapsto E^{\mathrm{s}} \oplus E^{\mathrm{c}}$ such that $W^u = \mathrm{graph}(h^{\mathrm{u}})$.*

Theorem 5.5 (Center manifold theorem)**.** *There is a neighborhood N of $x = 0$ and a C^{r-1} function $h^{\mathrm{c}} : N \cap E^{\mathrm{c}} \mapsto E^{\mathrm{u}} \oplus E^{\mathrm{s}}$ such that $\mathrm{graph}(h^{\mathrm{u}})$ is a W^{c}.*

Remark 5.6. One has that:

(a) $\dim(W^{\mathrm{s,c,u}}) = \dim(E^{\mathrm{s,c,u}})$

(b) The manifolds are invariant, i.e., if $x \in W^{\mathrm{s,c,u}}$, then $f^n(x) \in W^{\mathrm{s,c,u}}$ for all $n \in \mathbb{Z}$. Thus, if, e.g., $\dim(E^{\mathrm{s}}) = k_{\mathrm{s}}$, there is then a k_{s}-dimensional map which governs the behavior of the flow on W^{s}.

(c) $W^{\mathrm{s,c,u}}$ is tangent to $E^{\mathrm{s,c,u}}$ at $x = 0$.

(d) The dynamical behavior on W^{s} and W^{u} is determined solely by the linear behavior.

(e) W^{c} is not unique.

(f) The proofs of these theorems is discussed in [28, Chapter 3.3.5].

5.3. Examples

In order to fully illustrate the utility of the manifold theorems, consider the following examples.

5.3.1. Stable flow on W^{c}

Letting $x := (y,z)^{\mathrm{T}}$, consider

$$f(y,z) := \begin{pmatrix} \lambda y - z^2 \\ (1+y)z \end{pmatrix}, \quad \lambda \in (0,1).$$

For the fixed point $(0,0)^{\mathrm{T}}$ one has that

$$\sigma^{\mathrm{s}} = \{\lambda\},\ E^{\mathrm{s}} = \mathrm{span}\{(1,0)^{\mathrm{T}}\}; \quad \sigma^{\mathrm{c}} = \{1\},\ E^{\mathrm{c}} = \mathrm{span}\{(0,1)^{\mathrm{T}}\}.$$

There is a one-dimensional W^{s} and a one-dimensional W^{c}. What are the dynamics on W^{c}? The manifold W^{c} is given by the graph $y = h(z)$, where

$$h(z) = az^2 + \mathcal{O}(z^3).$$

Since W^{c} is invariant, upon using the equations one sees that

$$h((1+h(z))z) = \lambda h(z) - z^2,$$

which eventually implies that

$$h(z) = \frac{1}{\lambda - 1} z^2 + \mathcal{O}(z^3).$$

Thus, on dynamics on W^{c} are given by

$$z_{n+1} = G(z_n), \quad G(z) := z(1 + \frac{1}{\lambda-1} z^2 + \mathcal{O}(z^3)).$$

Since $\lambda \in (0,1)$ one has that $|G(z)| < |z|$ for $|z| \neq 0$ sufficiently small; hence, the fixed point $z = 0$ is asymptotically stable on W^{c}.

5.3.2. Hénon map

Letting $x := (w,z)^{\mathrm{T}}$, consider

$$f(w,z) := \begin{pmatrix} 1 + z - \lambda w^2 \\ 3w \end{pmatrix}.$$

The fixed points are given by $(w_\lambda^\pm, z_\lambda^\pm)^{\mathrm{T}}$, where

$$w_\lambda^\pm := \frac{1 \pm \sqrt{1+\lambda}}{\lambda}, \quad z_\lambda^\pm := 3w_\lambda^\pm.$$

When $\lambda = 3$ the fixed point $(w_3^-, z_3^-)^{\mathrm{T}} = (-1/3, -1)^{\mathrm{T}}$ is such that

$$\sigma^{\mathrm{u}} = \{3\},\ E^{\mathrm{u}} = \mathrm{span}\{(1,1)^{\mathrm{T}}\}; \quad \sigma^{\mathrm{c}} = \{-1\},\ E^{\mathrm{c}} = \mathrm{span}\{(1,-3)^{\mathrm{T}}\}.$$

Thus, at this fixed point there is an associated one-dimensional W^{u} and one-dimensional W^{c}.

What are the dynamics on W^{c}? As a consequence of the discussion in Section 4.1 one expects a pitchfork bifurcation to a period-2 orbit on W^{c}. In order to determine the stability of this orbit, one needs the equations which describe the dynamics on W^{c}. So that the behavior on W^{c} can be more easily analyzed, first put the system into canonical form via

$$\mu := \lambda - 3, \quad P := \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} := P^{-1} \begin{pmatrix} w + 1/3 \\ z + 1 \end{pmatrix}.$$

Under this transformation the system becomes

$$\begin{aligned} x_{n+1} &= \frac{1}{4}[-\frac{1}{9}\mu - 2(2 - \frac{1}{3}\mu)x_n + \frac{2}{3}\mu y_n - (3+\mu)(x_n + y_n)^2] \\ y_{n+1} &= \frac{1}{4}[-\frac{1}{3}\mu + 2\mu x_n + 6(2 + \frac{1}{3}\mu)y_n - 3(3+\mu)(x_n+y_n)^2]. \end{aligned}$$

The manifold W^{c} is given by the graph $y = h(x,\mu)$, where $h(0,0) = h_x(0,0) = 0$. Upon using the invariance of W^{c} it can eventually be shown that the dynamics on W^{c} are given by

$$x_{n+1} = G(x_n,\mu), \quad G(x,\mu) := -\frac{1}{36}\mu - (1 - \frac{4}{33}\mu)x - \frac{3}{4}x^2 - \frac{27}{16}x^3 + \mathcal{O}(x^4).$$

Since

$$G^2(x,\mu) = (1 - \frac{25}{88}\mu)x - \frac{9}{4}x^3 + \mathcal{O}(x^4),$$

upon following the analysis in Section 4.1 it is seen that on W^{c} there is a supercritical pitchfork bifurcation to an unstable period-2 orbit.

5.4. Homoclinic points

Much of the material discussed in this section is also discussed in [**28**, Chapters 23-24]. We will first discuss the dynamics associated with a relatively simple map. After this task has been accomplished, we will relate these dynamics to those associated with a seemingly more complicated problem.

5.4.1. The shift map

Let $A := \{0,1\}$, and let S denote the collection of all bi-infinite sequences of elements of A, i.e., if $s \in S$, then

$$s = \{\cdots s_{-n} \cdots s_{-2}s_{-1}.s_0s_1s_2\cdots s_n\cdots\}, \quad s_j \in A.$$

For $s, \hat{s} \in S$, define the distance between s and $\hat{s}$ via

$$d(s,\hat{s}) := \sum_{n=-\infty}^{+\infty} \frac{\delta_n}{2^{|n|}}, \quad \delta_n := \begin{cases} 0, & s_n = \hat{s}_n \\ 1, & s_n \neq \hat{s}_n, \end{cases}$$

and note that $d(s,\hat{s}) \le 3$. One then has that $d(s,\hat{s})$ is small if the two elements agree on a sufficiently long central block [**28**, Lemma 24.1.2]. Define the homeomorphism $\sigma : S \mapsto S$ by $\sigma(s)_n := s_{n+1}$, i.e.,

$$\sigma(s) = \{\cdots s_{-2}s_{-1}s_0.s_1s_2\cdots\}.$$

The mapping σ is known as the shift map.

Bi-infinite sequences which repeat periodically will be denoted by the finite length sequence with an overbar, i.e.,

$$\{\overline{01.01}\} = \{\cdots 010101.010101\cdots\}.$$

It is easy to see that

$$\sigma(\{\overline{0.0}\}) = \{\overline{0.0}\}, \quad \sigma(\{\overline{1.1}\}) = \{\overline{1.1}\};$$

hence, σ has two fixed points. In addition, one can check that

$$\sigma^2(\{\overline{10.10}\}) = \{\overline{10.10}\}.$$

Since

$$\sigma(\{\overline{10.10}\}) = \{\overline{01.01}\},$$

we have the existence of a period-2 orbit. It is clearly not difficult to construct period-n orbits for any $n \in \mathbb{N}$.

It will next be shown that σ has an uncountable number of nonperiodic orbits. Consider the bijective mapping between bi-infinite sequences and infinite sequences given by

$$\cdots s_{-n}\cdots s_{-2}s_{-1}.s_0s_1s_2\cdots s_n\cdots \mapsto .s_0s_1s_{-1}s_2s_{-2}s_3s_{-3}\cdots s_ns_{-n}\cdots.$$

Every $r \in [0,1]$ can be expressed in base 2 as a binary expansion via

$$r = \sum_{n=1}^{\infty} \frac{\delta_n}{2^n}, \quad \delta_n \in \{0,1\}.$$

If $r \in [0,1]\backslash\mathbb{Q}$, the sequence $.\delta_1\delta_2\delta_3\cdots$ is nonrepeating. Since $[0,1]\backslash\mathbb{Q}$ is uncountable, we now have the existence of an uncountable number of elements $s \in S$ such that $\sigma(s)$ is not a periodic orbit.

It will finally be shown that there is an element $s \in S$ whose orbit is dense. First construct all possible sequences having length $n \in \mathbb{N}$, e.g.,

$$\begin{aligned} &\text{length } 1: \quad \{0\},\ \{1\} \\ &\text{length } 2: \quad \{00\},\ \{01\},\ \{10\},\ \{11\}. \end{aligned}$$

For a given length n there will exist 2^n possible sequences. Order these sequences in the following manner. Let

$$s = \{s_1 \cdots s_k\}, \quad \hat{s} = \{\hat{s}_1 \cdots \hat{s}_j\}.$$

We say that $s < \hat{s}$ if $k < j$, and if $k = j$, then $s < \hat{s}$ if $s_\ell < \hat{s}_\ell$, where $\ell \in \mathbb{N}$ is the first such integer that $s_\ell \neq \hat{s}_\ell$. Denote the sequences having length k via $s_1^k < s_2^k < \cdots < s_{2^k}^k$. Now set

$$s_{\mathrm{d}} := \{\cdots s_8^3 s_6^3 s_4^3 s_2^3 s_4^2 s_2^2 s_2^1 . s_1^1 s_1^2 s_3^2 s_1^3 s_3^3 s_5^3 s_7^3 \cdots\},$$

and note that s_{d} contains all possible sequences of any fixed length. Let $s \in S$ be given, let $\epsilon > 0$ be given, and let $N \in \mathbb{N}$ be such that

$$\sum_{|j| \geq N} \frac{1}{2^{|j|}} < \epsilon.$$

Writing

$$s = \{\cdots s_{-N} \cdots s_{-1} . s_0 \cdots s_N \cdots\},$$

we have by construction that there exists an $M \in \mathbb{Z}$ such that $\sigma^M(s_{\mathrm{d}})_j = s_j$ for $|j| \leq N$, which implies that $d(\sigma^M(s_{\mathrm{d}}), s) < \epsilon$. Since $s \in S$ is arbitrary, as is $\epsilon > 0$, the orbit of s_{d} is dense in S.

Lemma 5.7. *The shift map $\sigma : S \mapsto S$ has*

(a) a countable infinity of periodic orbits of arbitrarily high period

(b) an uncountable infinity of nonperiodic orbits

(c) a dense orbit.

Remark 5.8. One has that:

(a) While we constructed one point which has a dense orbit, upon rearranging the ordering of the blocks s_j^k in s_{d} it can be clearly seen that there are at least a countable number of points which have a dense orbit.

(b) It can be shown that S is a closed, perfect (i.e., every point is a limit point), and totally disconnected set [**28**, Proposition 24.1.4]. In other words, S is a Cantor set.

(c) In the above we considered the shift map on two symbols. There is a discussion of symbolic dynamics on N symbols in [**28**, Chapter 24].

Let $s \in S$ be given, and let $\epsilon > 0$ be given. Let $\hat{s} \in S$ be such that $d(s, \hat{s}) < \epsilon$; furthermore, suppose that $s_j = \hat{s}_j$ for $j = -N, \dots, N$. Now suppose that $s_{N+1} \neq \hat{s}_{N+1}$. It is clear that

$$d(\sigma^{N+1}(s), \sigma^{N+1}(\hat{s})) \geq 1;$$

hence, for the given point s, and for each small neighborhood of s, there exists an uncountable number of points $\hat{s}$ such that after a fixed number of iterations these two points are separated by a fixed distance. A system displaying such behavior is said to exhibit sensitive dependence on initial conditions.

Definition 5.9. A dynamical system which displays sensitive dependence on initial conditions on a closed invariant set which consists of more than one orbit will be called chaotic.

5.4.2. Transverse homoclinic orbits

Definition 5.10. Let $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a smooth diffeomorphism with a hyperbolic fixed point p. If $W^{\mathrm{s}}(p) \cap W^{\mathrm{u}}(p) \neq \varnothing$, then we say that p is a transverse homoclinic point if for some $q \in W^{\mathrm{s}}(p) \cap W^{\mathrm{u}}(p)$

$$T_q W^{\mathrm{s}}(p) \oplus T_q W^{\mathrm{u}}(p) = \mathbb{R}^2.$$

Remark 5.11. One has that:

(a) For ODEs a homoclinic point cannot be transversal, as $\dot{x}_{\mathrm{h}} \in T_q W^{\mathrm{s}}(p) \cap T_q W^{\mathrm{u}}(p)$, where $x_{\mathrm{h}}(t)$ is the homoclinic orbit.

(b) If p is part of an orbit of period k, then one can discuss transverse homoclinic orbits for p simply by considering the mapping f^k.

Theorem 5.12. *Let $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a smooth diffeomorphism with a transverse homoclinic point p. For any neighborhood U of p there exists a Cantor set $\Lambda \subset U$ and an $n \in \mathbb{N}$ such that the mapping $f^n : \Lambda \mapsto \Lambda$ is topologically conjugate to the shift map on N symbols for some $N \geq 2$.*

Proof: See [**28**, Chapter 26] and [**9**, Chapter 5]. □

Remark 5.13. One has that:

(a) Since f^n is topologically equivalent to the shift map, in attempting to understand the dynamics for f^n it is sufficient to study those of the shift map.

(b) The set Λ is a hyperbolic invariant set.

Now suppose that $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is a smooth diffeomorphism, and that $p_0, p_1, \dots, p_n = p_0$ are hyperbolic fixed points. Further suppose that $W^{\mathrm{u}}(p_j) \cap W^{\mathrm{s}}(p_{j+1})$ transversely for $j = 0, \dots, n-1$. The fixed points, along with their stable and unstable manifolds, are then said to form a heteroclinic cycle. One can show that under this scenario $W^{\mathrm{u}}(p_j) \cap W^{\mathrm{s}}(p_j)$ transversely for each $j = 0, \dots, n-1$. Thus, as a consequence of Theorem 5.12 one has that there are invariant Cantor sets Λ_j in a neighborhood of p_j for $j = 0, \dots, n-1$ on which the dynamics are equivalent to those of the shift map.

5.4.3. Melnikov method revisited

We must now determine a concrete set of examples to which the result of Theorem 5.12 can be applied. Consider

$$\dot{x} = f(x) + \epsilon g(x,t), \tag{5.3}$$

where $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ and $g : \mathbb{R}^2 \times \mathbb{R} \mapsto \mathbb{R}^2$ are smooth, and $g(x, t+T) = g(x,t)$ for some $T > 0$. Assume that when $\epsilon = 0$,

(a) there exists hyperbolic critical point p_0 which is a saddle point

(b) there exists an orbit $q_0(t)$ homoclinic to p_0

(c) $\nabla \cdot f(x) \equiv 0$.

The last condition is automatically satisfied if the unperturbed system is Hamiltonian.

For each $t_0 \in [0, T)$ set

$$\Sigma_{t_0} := \{(x,t) : t = t_0\},$$

and note that there exists a Poincaré map $\Pi_{t_0} : \Sigma_{t_0} \mapsto \Sigma_{t_0}$ which is defined by the time-T map. When $\epsilon = 0$ one has that $\Pi_{t_0}(p_0) = p_0$; furthermore, since p_0 is a hyperbolic saddle point for each $\epsilon > 0$ there exists a $p_\epsilon = p_\epsilon(t_0)$ with $\lim_{\epsilon \to 0^+} p_\epsilon = p_0$ such that $\Pi_{t_0}(p_\epsilon) = p_\epsilon$. Finally, for each t_0 the point p_ϵ is a hyperbolic saddle point for Π_{t_0}, so that there exists a one-dimensional $W^{\mathrm{u}}(p_\epsilon)$ and $W^{\mathrm{s}}(p_\epsilon)$. Note that by supposition, when $\epsilon = 0$, $W^{\mathrm{u}}(p_0) \cap W^{\mathrm{s}}(p_0) \neq \varnothing$; however, the intersection is not transverse.

We wish to determine a condition which for $\epsilon > 0$ will not only guarantee that $W^{\mathrm{u}}(p_\epsilon) \cap W^{\mathrm{s}}(p_\epsilon) \neq \varnothing$, but that the intersection is transverse. Let $\gamma_\epsilon(t)$ be the T-periodic orbit which satisfies $\gamma_\epsilon(t_0) = p_\epsilon$. There exists stable and unstable manifolds for $\gamma_\epsilon(t)$ which are given by

$$W^{\mathrm{u}}(\gamma_\epsilon) := \bigcup_{t_0\in\mathbb{R}} W^{\mathrm{u}}(p_\epsilon), \quad W^{\mathrm{s}}(\gamma_\epsilon) := \bigcup_{t_0\in\mathbb{R}} W^{\mathrm{s}}(p_\epsilon).$$

Suppose that $q_0(t)$ intersects the line $x_2 = 0$ transversely. Now, $W^{\mathrm{u}}(\gamma_\epsilon) \cap \{x_2 = 0\}$ is given by the curve $x_1 = h^{\mathrm{u}}(t, \epsilon)$, and $W^{\mathrm{s}}(\gamma_\epsilon) \cap \{x_2 = 0\}$ is given by the curve $x_1 = h^{\mathrm{s}}(t, \epsilon)$. Upon setting

$$G(t, \epsilon) := h^{\mathrm{s}}(t, \epsilon) - h^{\mathrm{u}}(t, \epsilon),$$

one has that the desired transversal intersection will occur if and only if for some $t_0 \in [0, T)$,

$$G(t_0, \epsilon) = 0, \quad \frac{\partial}{\partial t} G(t_0, \epsilon) \neq 0. \tag{5.4}$$

Since $G(t, 0) \equiv 0$, one has that $G(t, \epsilon) = \epsilon \tilde{G}(t, \epsilon)$. The Melnikov function is given by

$$M(t) := \tilde{G}(t, 0).$$

As an application of the Implicit Function Theorem, one has that if

$$M(t_0) = 0, \quad M'(t_0) \neq 0, \tag{5.5}$$

then equation (5.4) is satisfied; hence, if equation (5.5) is satisfied, the relevant manifolds intersect transversely, and the result of Theorem 5.12 applies to Poincaré map associated with equation (5.3). In a manner similar to that which led to Theorem 3.2, it can be shown that the Melnikov function is given by

$$M(t_0) := \int_{-\infty}^{+\infty} d(f(q_0(t)), g(q_0(t), t + t_0))\,\mathrm{d}t \tag{5.6}$$

(also see [**28**, Chapter 28] and [**9**, Chapter 4.5]).

Remark 5.14. The interested student should compare equation (5.6) with equation (3.8).

5.4.4. Example: The forced, damped Duffing oscillator

The forced, damped Duffing oscillator is given by

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x - x^3 + \epsilon(\gamma \cos \omega t - \delta y). \end{aligned} \tag{5.7}$$

When $\epsilon = 0$ there exist homoclinic orbits $q_0^{\pm}(t)$ which are given by

$$q_0^{\pm}(t) = \begin{pmatrix} x_0^{\pm}(t) \\ y_0^{\pm}(t) \end{pmatrix} := \sqrt{2} \begin{pmatrix} \pm \operatorname{sech} t \\ \mp \operatorname{sech} t \tanh t \end{pmatrix}.$$

Upon using equation (5.6) one sees that the Melnikov function for each orbit is given by

$$\begin{aligned} M^{\pm}(t_0) &= -\delta \int_{-\infty}^{+\infty} (y_0^{\pm}(t))^2\,\mathrm{d}t \pm \gamma \int_{-\infty}^{+\infty} y_0^{\pm}(t) \cos \omega(t + t_0)\,\mathrm{d}t \\ &= \frac{4}{3}(-\delta \pm \gamma R(\omega) \sin \omega t_0), \qquad R(\omega) := \frac{3}{2\sqrt{2}} \pi\omega \operatorname{sech}(\frac{\pi\omega}{2}). \end{aligned}$$

A plot of critical surface $\delta = R(\omega)\gamma$ is given in [**28**, Figure 28.5.2]. One has that if $|\delta/\gamma| < R(\omega)$, then there exists a $t_0 \in [0, 2\pi/\omega)$ such that $M(t_0) = 0$ with $M'(t_0) \neq 0$ (see [**28**, Figure 28.5.1]). Hence, in this case there are chaotic dynamics associated with equation (5.7). Conversely, if $|\delta/\gamma| > R(\omega)$, then $M(t_0) \neq 0$ for all $t_0 \in [0, 2\pi/\omega)$, so that the manifolds do not intersect. Hence, in this case one cannot conclude the existence of chaotic dynamics.

6. Method of Averaging

Most of the material considered in this section can be found in [**9**, Chapter 4] (also see [**26**]). Consider the weakly forced nonlinear oscillator given by

$$\ddot{x} + \omega_0^2 x = \epsilon f(x, \dot{x}, t), \tag{6.1}$$

where $f(x, \dot{x}, t + T) = f(x, \dot{x}, t) : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is smooth, and $0 \le \epsilon \ll 1$. Suppose that $T = 2\pi/\omega$ for some $\omega \in \mathbb{R}^+$. For a given $k \in \mathbb{N}$, set

$$A(t) := \begin{pmatrix} \cos\phi & -(k/\omega)\sin\phi \\ -\sin\phi & -(k/\omega)\cos\phi \end{pmatrix}, \quad \phi := \frac{\omega t}{k},$$

and define

$$\begin{pmatrix} u \\ v \end{pmatrix} := A(t) \begin{pmatrix} x \\ \dot{x} \end{pmatrix}. \tag{6.2}$$

Note that $A(t)$ is periodic with period $2\pi k/\omega$. Under this transformation equation (6.1) becomes

$$\begin{aligned} \dot{u} &= -\frac{k}{\omega}\left[\left(\frac{\omega^2 - k^2\omega_0^2}{k^2}\right)x + \epsilon f(x, \dot{x}, t)\right]\sin\phi \\ \dot{v} &= -\frac{k}{\omega}\left[\left(\frac{\omega^2 - k^2\omega_0^2}{k^2}\right)x + \epsilon f(x, \dot{x}, t)\right]\cos\phi. \end{aligned} \tag{6.3}$$

In equation (6.3) one has that $x = x(u, v)$ and $\dot{x} = \dot{x}(u, v)$, where these functions are determined by inverting the relation in equation (6.2). Note that if

$$\omega^2 - k^2\omega_0^2 = \mathcal{O}(\epsilon),$$

i.e., the system is close to a resonance of order k, then equation (6.3) is an example of the general system

$$\dot{x} = \epsilon f(x, t, \epsilon), \tag{6.4}$$

where $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}^n$ is smooth and of period $T > 0$ in t.

6.1. Averaging

Regarding equation (6.4), set

$$f(x, t, \epsilon) = \bar{f}(x) + \tilde{f}(x, t, \epsilon), \quad \bar{f}(x) := \frac{1}{T}\int_0^T f(y, t, 0)\,\mathrm{d}y.$$

Here $\bar{f}$ is the mean of f, and $\tilde{f}$ is the oscillating part of f. Now set

$$x := y + \epsilon w(y, t, \epsilon), \tag{6.5}$$

where w has yet to be determined. Equation (6.5) is an example of a near-identity transformation. Differentiating equation (6.5) and using equation (6.4) yields that

$$\begin{aligned} (\mathbb{1} + \epsilon \mathrm{D}_y w)\dot{y} &= \dot{x} - \epsilon\frac{\partial w}{\partial t} \\ &= \epsilon\bar{f}(y + \epsilon w) + \epsilon\tilde{f}(y + \epsilon w, t, \epsilon) - \epsilon\frac{\partial w}{\partial t}. \end{aligned}$$

Since

$$(\mathbb{1} + \epsilon \mathrm{D}_y w)^{-1} = \mathbb{1} - \epsilon \mathrm{D}_y w + \mathcal{O}(\epsilon^2),$$

upon doing a Taylor expansion for $\bar{f}$ and $\tilde{f}$ one sees that

$$\dot{y} = \epsilon(\bar{f}(y) + \tilde{f}(y,t,0) - \frac{\partial w}{\partial t}) + \epsilon^2 f_1(y,t) + \mathcal{O}(\epsilon^3),$$

where

$$f_1(y,t) := \mathrm{D}_y f(y,t,0)w(y,t,0) - \mathrm{D}_y w(y,t,0)\bar{f}(y) + \frac{\partial \tilde{f}}{\partial \epsilon}(y,t,0). \tag{6.6}$$

If one sets

$$\frac{\partial w}{\partial t} = \tilde{f}(y,t,0), \tag{6.7}$$

then one finally arrives at the system

$$\dot{y} = \epsilon \bar{f}(y) + \epsilon^2 f_1(y,t) + \mathcal{O}(\epsilon^3). \tag{6.8}$$

Note that $f_1(y, t+T) = f_1(y,t)$; hence, equation (6.8) can be thought of as a periodically forced autonomous system.

Theorem 6.1 (Averaging Theorem)**.** *Consider equation (6.4), and the associated averaged system*

$$\dot{y} = \epsilon \bar{f}(y). \tag{6.9}$$

The following are true:

(a) If $x(t)$ is a solution to equation (6.4) and $y(t)$ is a solution to equation (6.9), and if $|x(0) - y(0)| = \mathcal{O}(\epsilon)$, then $|x(t) - y(t)| = \mathcal{O}(\epsilon)$ on a time scale $t \sim \epsilon^{-1}$.

(b) If p_0 is a hyperbolic fixed point for equation (6.9), then for $\epsilon > 0$ sufficiently small equation (6.4) possesses a unique hyperbolic periodic orbit $\gamma_\epsilon(t) = p_0 + \mathcal{O}(\epsilon)$ of the same stability type as p_0.

(c) If p_0 is a hyperbolic fixed point, and if $x_{\mathrm{s}}(t) \in W^{\mathrm{s}}(\gamma_\epsilon)$ and $y_{\mathrm{s}}(t) \in W^{\mathrm{s}}(p_0)$, then $|x_{\mathrm{s}}(0) - y_{\mathrm{s}}(0)| = \mathcal{O}(\epsilon)$ implies that $|x_{\mathrm{s}}(t) - y_{\mathrm{s}}(t)| = \mathcal{O}(\epsilon)$ for all $t \geq 0$.

(d) If p_0 is a hyperbolic fixed point, and if $x_{\mathrm{u}}(t) \in W^{\mathrm{u}}(\gamma_\epsilon)$ and $y_{\mathrm{u}}(t) \in W^{\mathrm{u}}(p_0)$, then $|x_{\mathrm{u}}(0) - y_{\mathrm{u}}(0)| = \mathcal{O}(\epsilon)$ implies that $|x_{\mathrm{u}}(t) - y_{\mathrm{u}}(t)| = \mathcal{O}(\epsilon)$ for all $t \leq 0$.

Proof: Let $y_\epsilon(t)$ denote the solution to equation (6.8), and let $y_{\mathrm{avg}}(t)$ denote the solution to equation (6.9). If $x(t)$ is the solution to equation (6.4), then via the transformation in equation (6.5) one has that

$$|x(t) - y_\epsilon(t)| = \epsilon |w(y_\epsilon, t, \epsilon)| = \mathcal{O}(\epsilon). \tag{6.10}$$

The estimate follows from the fact that

$$\int_0^T \tilde{f}(y,t,\epsilon)\,\mathrm{d}t = 0.$$

Thus, upon using the triangle inequality

$$|x(t) - y_{\mathrm{avg}}(t)| \leq |x(t) - y_\epsilon(t)| + |y_\epsilon(t) - y_{\mathrm{avg}}(t)|,$$

the result of part (a) is proved if $|y_\epsilon(t) - y_{\mathrm{avg}}(t)| = \mathcal{O}(\epsilon)$ for $t \sim \epsilon^{-1}$.

Now consider equation (6.8) and equation (6.9), and recall that the solution to each is given by an integral equation. Letting $y_{\mathrm{d}} := y_\epsilon - y_{\mathrm{avg}}$, L be the Lipschitz constant of $\bar{f}$, and C the maximum value of f_1, and subtracting the two solutions eventually yields

$$|y_{\mathrm{d}}(t)| \leq |y_{\mathrm{d}}(0)| + \epsilon^2 Ct + \epsilon L \int_0^t |y_{\mathrm{d}}(s)|\,\mathrm{d}s.$$

Applying a generalized Gronwall's inequality (see [**9**, Lemma 4.1.2]) yields that

$$\begin{aligned}|y_{\mathrm{d}}(t)| &\le |y_{\mathrm{d}}(0)|\mathrm{e}^{\epsilon Lt} + \epsilon^2 C\int_0^t \mathrm{e}^{\epsilon L(t-s)}\,\mathrm{d}s\\ &\le \left(|y_{\mathrm{d}}(0)| + \epsilon\frac{C}{L}\right)\mathrm{e}^{\epsilon Lt}.\end{aligned}$$

Thus, if $|y_{\mathrm{d}}(0)| = \mathcal{O}(\epsilon)$, then one can conclude that $|y_{\mathrm{d}}(t)| = \mathcal{O}(\epsilon)$ for $0 \le t \le (\epsilon L)^{-1}$.

For part (b), the general idea is to construct a Poincaré map for equation (6.8) and equation (6.9), and then show that the fixed points and manifolds for each are within $\mathcal{O}(\epsilon)$. The change of variables in equation (6.10) is used implicitly throughout. Let Π_0 (Π_ϵ) denote the time-T Poincaré map for equation (6.9) (equation (6.8)). Note that Π_ϵ is within $\mathcal{O}(\epsilon^2)$ to Π_0, since T is fixed independent of ϵ. Set

$$g_{0,\epsilon}(p,\epsilon) := \frac{1}{\epsilon}(\Pi_{0,\epsilon}(p) - p),$$

and note that for each $\epsilon > 0$ one has that $g_0(p_0,\epsilon) = 0$. Furthermore, since $\mathrm{D}_p\Pi_0(p_0) = \mathrm{e}^{\epsilon T\mathrm{D}\bar{f}(p_0)}$, one has that

$$\lim_{\epsilon\to0^+} \mathrm{D}_p g_0(p_0,\epsilon) = T\mathrm{D}\bar{f}(p_0),$$

which is nonsingular. Now, since Π_ϵ is ϵ^2-close to Π_0, one has that

$$\lim_{\epsilon\to0^+} g_\epsilon(p_0,\epsilon) = 0, \quad \lim_{\epsilon\to0^+} \mathrm{D}_p g_\epsilon(p_0,\epsilon) = T\mathrm{D}\bar{f}(p_0).$$

Hence, upon applying the Implicit Function Theorem one has a unique curve of points p_ϵ which are within $\mathcal{O}(\epsilon)$ of p_0 that satisfy $g_\epsilon(p_\epsilon,\epsilon) \equiv 0$. Furthermore, since $p_\epsilon = p_0 + \mathcal{O}(\epsilon)$ one has that

$$\mathrm{D}_p\Pi_\epsilon(p_\epsilon) = \mathrm{e}^{\epsilon T\mathrm{D}\bar{f}(p_0)} + \mathcal{O}(\epsilon^2),$$

so that the fixed point p_ϵ is also hyperbolic.

The proof for parts (c) and (d) is given in [**9**, Chapter 4.1]. □

Remark 6.2. One has that:

(a) Parts (c) and (d) of the averaging theorem can be paraphrased to say that $W^{\mathrm{s,u}}(p_0)$ approximate to $\mathcal{O}(\epsilon)$ the stable and unstable manifolds of the Poincaré map of the full system in equation (6.4).

(b) The persistence of the critical point as a periodic orbit is guaranteed via the Implicit Function Theorem as long as $\lambda = 0 \notin \sigma(\mathrm{D}\bar{f}(p_0))$. However, in the case that $\mathrm{Re}\,\lambda = 0$ one cannot make a conclusion regarding the stability of the fixed point for the Poincaré map without further analysis.

Remark 6.3. In some cases second-order, or even higher-order, averaging may be required. In such a case, upon setting

$$\bar{f}_1(z) := \frac{1}{T}\int_0^T f_1(z,t,0),\mathrm{d}t,$$

and using the second transformation

$$y := z + \epsilon^2 w(z,t,\epsilon),$$

one finds that equation (6.4) becomes

$$\dot{z} = \epsilon\bar{f}(z) + \epsilon^2\bar{f}_1(z) + \mathcal{O}(\epsilon^3).$$

The second-order averaged equation is then

$$\dot{z} = \epsilon\bar{f}(z) + \epsilon^2\bar{f}_1(z).$$

One can show that the solutions are a good approximation on a time scale of $\mathcal{O}(\epsilon^{-2})$.

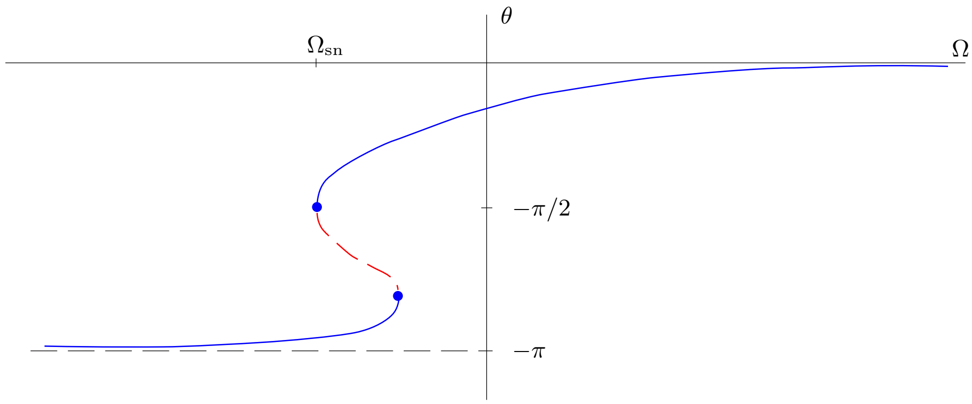

Figure 1: The zero set of $g_{\rm c}(\theta)$. The solid line corresponds to stable solutions, while the dashed line corresponds to saddle solutions.

6.1.1. Example: Duffing equation

Consider equation (6.1) with

$$f(x,\dot{x},t)=\gamma\cos\omega t-\delta\dot{x}-\alpha x^3,$$

and suppose that

$$\omega_0^2-\omega^2=\epsilon\Omega.$$

All of the parameters are assumed to be positive. Setting $k=1$ in equation (6.3) eventually yields the system

$$\begin{aligned}\dot{u}&=\frac{\epsilon}{\omega}[\Omega(u\cos\omega t-v\sin\omega t)-\omega\delta(u\sin\omega t+v\cos\omega t)\\&\quad+\alpha(u\cos\omega t-v\sin\omega t)^3-\gamma\cos\omega t]\sin\omega t\\\dot{v}&=\frac{\epsilon}{\omega}[\Omega(u\cos\omega t-v\sin\omega t)-\omega\delta(u\sin\omega t+v\cos\omega t)\\&\quad+\alpha(u\cos\omega t-v\sin\omega t)^3-\gamma\cos\omega t]\cos\omega t.\end{aligned}\tag{6.11}$$

The averaged equation ($T=2\pi/\omega$) associated with equation (6.11) is given in polar coordinates by

$$\begin{aligned}\dot{r}&=\frac{\epsilon}{2\omega}(-\delta\omega r-\gamma\sin\theta)\\r\dot{\theta}&=\frac{\epsilon}{2\omega}\left(\Omega r+\frac{3}{4}\alpha r^3-\gamma\cos\theta\right).\end{aligned}\tag{6.12}$$

The critical points for equation (6.12) are given by

$$r=-\frac{\gamma}{\delta\omega}\sin\theta,\quad g_{\rm c}(\theta)=0,$$

where

$$g_{\rm c}(\theta):=\Omega(\delta\omega)^2\sin\theta+\frac{3}{4}\alpha\gamma^2\sin^3\theta+(\delta\omega)^3\cos\theta.$$

Note that $r\ge 0$ necessarily implies that the only relevant critical points satisfy $\sin\theta\le 0$. Set

$$a:=\delta\omega,\quad b:=\frac{3}{4}\alpha\gamma^2.$$

One solution to $g_{\rm c}(\theta)=0$ is given by

$$(\Omega_{\rm sn},\theta_{\rm sn}):=(-\frac{b}{a^2},-\frac{\pi}{2}).$$

A Taylor expansion of g_c about $(\Omega_{sn}, \theta_{sn})$ yields the normal form

$$g_n(\Omega, \theta) := -a^2(\Omega - \Omega_{sn}) + b(\theta - \theta_{sn})^2;$$

hence, a saddle-node bifurcation occurs at $(\Omega_{sn}, \theta_{sn})$. Upon using the fact that for $|\Omega| \gg 1$ one has that $g_c(\theta) = 0$ for $\theta \sim 0 \pmod{\pi}$, one gets the diagram for the critical points seen in Figure 1 (also see [**9**, Figure 4.2.1]). The stability types of the branches of the steady solutions are obtained by consideration of the eigenvalues of the linearized averaged equation (6.12). Since all of the critical points are hyperbolic, they persist as periodic orbits for equation (6.11)

Sample phase portraits for equation (6.12) in the case that $\delta \neq 0$ are given in [**9**, Figure 4.2.2], and a comparison of the averaged flow with the Poincaré map for equation (6.11) is given in [**9**, Figure 4.2.3]. If one writes equation (6.12) in Cartesian coordinates, then an easy calculation shows that

$$\operatorname{trace} \mathrm{D}\bar{f} = -\delta\epsilon < 0,$$

so that an application of Bendixson's criterion yields that there exist no periodic orbits.

If $\delta = 0$, the original system is Hamiltonian, and the Hamiltonian for the averaged flow is given by

$$\bar{H}(u, v) := -\frac{\epsilon}{4\omega}\left[\Omega(u^2 + v^2) + \frac{3}{8}\alpha(u^2 + v^2)^2 - 2\gamma u\right].$$

The phase portrait for the averaged system is given in [**9**, Figure 4.4.1]. Since the periodic orbits are not hyperbolic, one cannot conclude that there exist invariant tori for equation (6.12) (see Theorem 6.6). This question is left open for further study.

6.1.2. Example: Bose-Einstein condensates

Recall equation (1.5),

$$\begin{aligned}
\dot{\rho}_0 &= -2\delta R^2 \rho_0(1 - \rho_0)\sin 2\Delta\phi \\
\dot{\Delta\phi} &= -\frac{1}{2}(\mu_1 - \mu_0) - \epsilon(\alpha_{11}^{\mathrm{p}} - \alpha_{00}^{\mathrm{p}})p(t) \\
&\quad + \delta R^2\left(\alpha_{111}^1 - (\alpha_{000}^0 + \alpha_{111}^1)\rho_0 + \alpha_{011}^0(2 + \cos 2\Delta\phi)(2\rho_0 - 1)\right),
\end{aligned} \tag{6.13}$$

where $p(t + T) = p(t)$ for some $T \in \mathbb{R}^+$. Upon setting

$$\Delta\mu := \mu_1 - \mu_0, \quad \psi := \Delta\phi + \frac{1}{2}\Delta\mu t,$$

equation (6.13) can be rewritten as

$$\begin{aligned}
\dot{\rho}_0 &= -2\delta R^2 \rho_0(1 - \rho_0)\left[\sin 2\psi \cos \Delta\mu t - \cos 2\psi \sin \Delta\mu t\right] \\
\dot{\psi} &= -\epsilon(\alpha_{11}^{\mathrm{p}} - \alpha_{00}^{\mathrm{p}})p(t) + \delta R^2\left(\alpha_{111}^1 - 2\alpha_{011}^0 + (4\alpha_{011}^0 - \alpha_{000}^0 - \alpha_{111}^1)\rho_0\right. \\
&\quad \left. + \alpha_{011}^0(2\rho_0 - 1)\left[\sin 2\psi \sin \Delta\mu t + \cos 2\psi \cos \Delta\mu t\right]\right).
\end{aligned} \tag{6.14}$$

Note that under that assumption that $p(t)$ is even one has that equation (6.14) is invariant under the actions

$$(\rho_0, \psi, t) \mapsto (\rho_0, -\psi, -t), \quad (\rho_0, \psi, t) \mapsto (\rho_0, \psi + \pi, t). \tag{6.15}$$

If one now assumes that

$$R = A\epsilon^{1/2}; \quad T = \frac{2\pi}{k\Delta\mu}, \ k \in \mathbb{N}, \tag{6.16}$$

then the results of Theorem 6.1 apply to equation (6.14). Writing

$$p(t) = p_{\mathrm{av}} + \tilde{p}(t), \quad \int_0^T \tilde{p}(t)\,\mathrm{d}t = 0,$$

yields that the averaged system is given by

$$\begin{aligned}\dot{\bar\rho}_0 &= 0\\ \dot{\bar\psi} &= -\epsilon k(\alpha_{11}^{\mathrm{p}}-\alpha_{00}^{\mathrm{p}})p_{\mathrm{av}}+\delta\epsilon A^2\left(\alpha_{111}^1-2\alpha_{011}^0+(4\alpha_{011}^0-\alpha_{000}^0-\alpha_{111}^1)\bar\rho_0\right).\end{aligned}\tag{6.17}$$

The dynamics associated with equation (6.17) are relatively uninteresting, and hence it will be considered no longer.

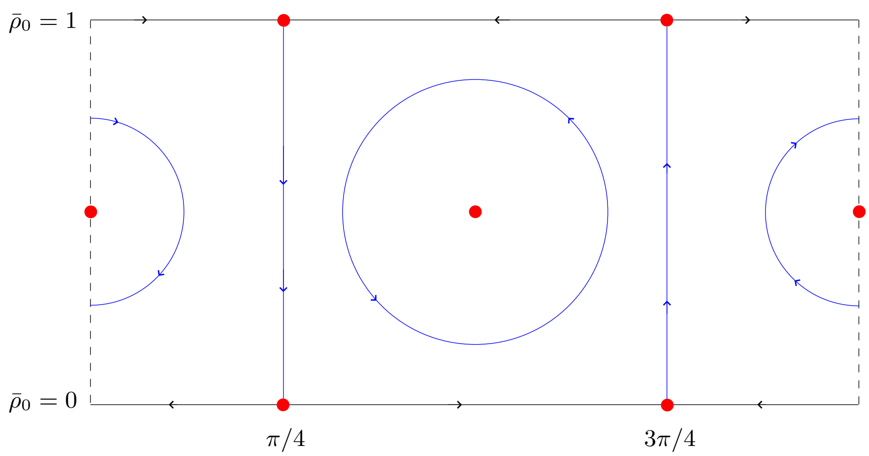

Figure 2: The phase portrait associated with the averaged equation (6.18).

Now consider equation (6.14) in the event that $\epsilon = 0$, and that $\delta(t+T)=\delta(t)$ for some $T\in\mathbb{R}^+$. This case arises physically in the presence of a Feshbach resonance [**22**, **23**]. Assuming equation (6.16) with $A = 1$, one can again apply Theorem 6.1 to equation (6.14). For ease, now assume that $\delta(t)$ is even, so that

$$\delta(t)=\sum_{k=0}^{\infty}\delta_k\cos(k\Delta\mu t).$$

Note that this assumption implies that the invariances associated with equation (6.15) still hold. If $\delta_0 = 0$ the averaged system is

$$\begin{aligned}\dot{\bar\rho}_0 &= -\epsilon\delta_1\bar\rho_0(1-\bar\rho_0)\sin 2\bar\psi\\ \dot{\bar\psi} &= \epsilon\delta_1\alpha_{011}^0\left(\bar\rho_0-\frac{1}{2}\right)\cos 2\bar\psi.\end{aligned}\tag{6.18}$$

Note that if $\delta_1 = 0$, then equation (6.18) reduces to the trivial system

$$\dot{\bar\rho}_0 = 0,\quad \dot{\bar\psi} = 0;$$

hence, if $\delta_1 \neq 0$ the system is being forced at resonance. The solutions to equation (6.18) satisfy

$$[\bar\rho_0(1-\bar\rho_0)]^{\alpha_{011}^0}\cos 2\bar\psi = C.$$

If $\alpha_{011}^0 = 1$, then equation (6.18) would be a Hamiltonian system; however, an application of the Hölder inequality shows that

$$\alpha_{011}^0 \le \min\left\{\sup_x|q_0(x)|^2,\sup_x|q_1(x)|^2\right\},$$

and as such one generically has that $\alpha_{011}^0 < 1$.

Remark 6.4. If $\delta_0 \neq 0$, then the term

$$\epsilon\delta_0 \left(\alpha_{111}^1 - 2\alpha_{011}^0 + (4\alpha_{011}^0 - \alpha_{000}^0 - \alpha_{111}^1)\bar{\rho}_0\right)$$

is added to the equation for $\dot{\bar{\psi}}$ in equation (6.18). Thus, δ_0 can be used as an unfolding parameter.

The critical points satisfying $\bar{\rho}_0 \in \{0,1\}$, $\bar{\psi} \in \{\pi/4, 3\pi/4\}$ are hyperbolic, and as a consequence persist as periodic orbits for equation (6.13). The critical points $\bar{\rho}_0 = 1/2$, $\bar{\psi} \in \{0, \pi/2\}$ are nonlinear centers, and the eigenvalues of the linearization are given by

$$\lambda = \pm\mathrm{i}\left(\frac{\alpha_{011}^0}{2}\right)^{1/2}\epsilon.$$

These critical points also persist as periodic orbits for equation (6.13). The phase portrait for equation (6.18) is given in Figure 2. Since the periodic orbits are not hyperbolic, one cannot conclude that there exist invariant tori for equation (6.13) (see Theorem 6.6). This question is left open for further study.

6.2. Local bifurcations, global behavior, and Hamiltonian systems

One can prove the following theorems via an analysis of the Poincaré map as in the proof of Theorem 6.1. The details can be found in [**9**, Chapter 4.3-4]. It should be noted that if a Poincaré map undergoes a Hopf bifurcation, then that implies that for the map there exists an invariant closed curve γ; however, the dynamics of the Poincaré map on γ do not need to be periodic.

Theorem 6.5. *If equation (6.9) undergoes a saddle-node or Hopf bifurcation, the for $\epsilon > 0$ and sufficiently small the Poincaré map associated with equation (6.4) also undergoes a saddle-node or a Hopf bifurcation.*

Theorem 6.6. *If there exists a hyperbolic periodic orbit γ_0 for equation (6.9), then for the Poincaré map associated with equation (6.4) there exists an invariant closed curve γ_ϵ which is close to γ_0.*

If the original system equation (6.4) is Hamiltonian, then the transformation leading to equation (6.8) can be chosen so that the system remains Hamiltonian [**8**, Chapter 9]. In particular, the averaged system equation (6.9) is Hamiltonian. If $x \in \mathbb{R}^2$, the solution curves are the level curves of the averaged Hamiltonian. If there exists a homoclinic or heteroclinic orbit for the unperturbed Poincaré map, the relevant manifolds necessarily intersect in a nontransverse fashion; hence, we cannot expect it to be preserved for the system equation (6.8).

6.3. Comparison with a multiple time scales expansion

The bulk of the material presented herein can be found in [**26**, Chapter 8.2]. Again consider equation (6.4), and assume that for some $N \in \mathbb{N}$ the solution has the form

$$x(t,\tau) = \sum_{j=0}^{N} \epsilon^j x_j(t,\tau), \quad \tau := \epsilon t. \tag{6.19}$$

By the chain rule one has that

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \epsilon\frac{\partial}{\partial \tau},$$

so that equation (6.4) can be rewritten as

$$\frac{\partial x_0}{\partial t} + \epsilon\left(\frac{\partial x_0}{\partial \tau} + \frac{\partial x_1}{\partial t}\right) + \epsilon^2\left(\frac{\partial x_1}{\partial \tau} + \frac{\partial x_2}{\partial t}\right) + \mathcal{O}(\epsilon^3) = \epsilon f(x_0 + \epsilon x_1 + \mathcal{O}(\epsilon^2), t, \epsilon). \tag{6.20}$$

Since

$$f(x_0 + \epsilon x_1 + \cdots, t, \epsilon) = f(x_0, t, 0) + \epsilon\left(\mathrm{D}_x f(x_0,t,0)x_1 + \frac{\partial f}{\partial \epsilon}(x_0,t,0)\right) + \mathcal{O}(\epsilon^2),$$

upon equating like powers of ϵ in equation (6.20) one arrives at the sequence of equations

$$\begin{aligned}\frac{\partial x_0}{\partial t} &= 0\\ \frac{\partial x_1}{\partial t} &= -\frac{\partial x_0}{\partial \tau} + f(x_0,t,0)\\ \frac{\partial x_2}{\partial t} &= -\frac{\partial x_1}{\partial \tau} + \mathrm{D}_x f(x_0,t,0)x_1 + \frac{\partial f}{\partial \epsilon}(x_0,t,0),\end{aligned} \tag{6.21}$$

which can be solved successively.

Integrating the first equation in equation (6.21) yields

$$x_0(t,\tau) = A(\tau),$$

and integrating the second equation in equation (6.21) yields

$$x_1(t,\tau) = \int_0^t \left[-\frac{\partial A}{\partial \tau} + f(A(\epsilon s),s,0)\right] \mathrm{d}s + B(\tau), \quad B(0) = 0. \tag{6.22}$$

In order that terms do not become unbounded, we must now enforce the secularity condition that

$$\int_0^t \left[-\frac{\mathrm{d}A}{\mathrm{d}\tau} + f(A(\epsilon s),s,0)\right] \mathrm{d}s = 0,$$

i.e.,

$$\frac{\mathrm{d}A}{\mathrm{d}\tau} = \frac{1}{T}\int_0^T f(A,s,0)\,\mathrm{d}s, \tag{6.23}$$

i.e.,

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \epsilon\bar{f}(A).$$

Note that equation (6.23) is exactly the averaged equation (6.9). Further note that equation (6.22) can then be rewritten as

$$x_1(t,\tau) = \int_0^t \left[f(A(\epsilon s),s,0) - \bar{f}(A(\epsilon s))\right] \mathrm{d}s + B(\tau), \quad B(0) = 0.$$

In order to determine $B(\tau)$, one proceeds as above. Plugging the solutions for x_0 and x_1 into the nonhomogeneous equation for x_2 in equation (6.21) and applying another secularity condition eventually yields the nonhomogeneous linear system

$$\frac{\mathrm{d}B}{\mathrm{d}\tau} = \mathrm{D}_x\bar{f}(A)B(\tau) + \frac{1}{T}\int_0^T \left[-\frac{\partial}{\partial \tau} + \mathrm{D}_x f(A,t,0)\right](x_1(t,\tau) - B(\tau))\,\mathrm{d}t.$$

It can be shown that the expansion in equation (6.19), where x_0 and x_1 are prescribed as above, yields an $\mathcal{O}(\epsilon^2)$ approximation on the time scale $t \sim \epsilon^{-1}$ [**26**, Theorem 8.2.1].

Remark 6.7. It is important to note here that the $\mathcal{O}(\epsilon)$ correction to the averaged equation operates on both time scales. In particular, $B(\tau)$ adds an $\mathcal{O}(\epsilon)$ correction on the time scale $t \sim \epsilon^{-1}$, and hence its evaluation can be considered to be relatively unimportant.

For a simple example, consider

$$\dot{x} = \epsilon(x - x^2)\cos^2 t. \tag{6.24}$$

By equation (6.23) one sees that

$$\frac{\mathrm{d}A}{\mathrm{d}\tau} = \frac{1}{2}A(1-A), \quad A(0) = A_0,$$

i.e.,

$$A(\tau) = \frac{A_0}{A_0 + (1-A_0)\mathrm{e}^{-\tau/2}}.$$

Now, upon setting

$$
\begin{aligned}
u_1(t,\tau) &:= \int_0^t \left[f(A(\epsilon s), s, 0) - \bar{f}(A(\epsilon s)) \right] \mathrm{d}s \\
&= \frac{1}{2}\int_0^t \frac{\mathrm{d}A}{\mathrm{d}\tau}\cos 2s \,\mathrm{d}s \\
&= \frac{1}{4}A_0(1-A_0)\int_0^t \frac{\mathrm{e}^{-\epsilon s/2}}{(A_0+(1-A_0)\mathrm{e}^{-\epsilon s/2})^2}\cos 2s\,\mathrm{d}s,
\end{aligned}
$$

one has that

$$x_1(t,\tau) = u_1(t,\tau) + B(\tau),$$

where $B(\cdot)$ satisfies the ODE

$$\frac{\mathrm{d}B}{\mathrm{d}\tau} = (\frac{1}{2} - A)B + \frac{1}{T}\int_0^T \left[-\frac{\partial}{\partial\tau} + (\frac{1}{2} - A)\cos 2s \right] u_1(t,\tau)\,\mathrm{d}t, \quad B(0) = 0.$$

It can be seen in Figure 3 that $u_1(t,\tau)$ is oscillatory on the time scale $t = \mathcal{O}(1)$, and is slowly growing on the time scale $t = \mathcal{O}(\epsilon^{-1})$.

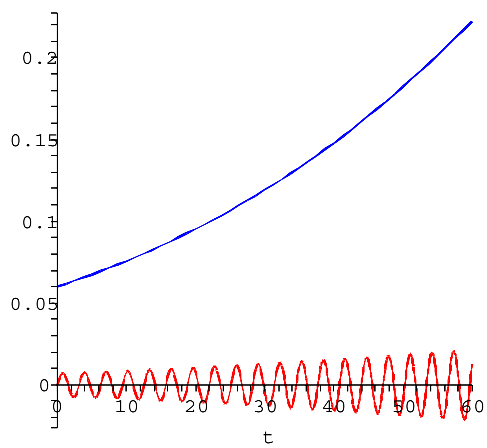

Figure 3: A graph of both the averaged solution $A(\tau)$ (blue curve) and the first-order "fast" correction $u_1(t,\tau)$ (red curve) for equation (6.24). Here $\epsilon = 0.05$ and $A_0 = 0.06$.

6.4. Almost-periodic vector fields

When discussing the Feshbach resonance in Section 6.1.2, in order to apply the theory of averaging we were required to make the somewhat artificial assumption on the period of the resonant forcing. We will now generalize Theorem 6.1 so that this restriction can be removed. A complete discussion of almost-periodic differential equations can be found in [**7**].

6.4.1. Preliminary estimates

Consider an almost periodic function $p(t) : \mathbb{R} \mapsto \mathbb{R}$ which can be written as

$$p(t) = \sum_{j=-\infty}^{+\infty} p_j \mathrm{e}^{\mathrm{i}\lambda_j t}, \tag{6.25}$$

where $\lambda_j \in \mathbb{R}$ and

$$p_j = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} p(t)\mathrm{e}^{-\mathrm{i}\lambda_j t}\,\mathrm{d}t.$$

If $p(t)$ were periodic, then one would have that $\lambda_j = 2\pi\alpha j$ for some $\alpha \in \mathbb{R}^+$. Now assume that $\lambda_0 = 0$, and that $\lambda_j \neq 0$ for $j \in \mathbb{Z}\backslash\{0\}$. Further assume that $\lambda_j \neq \lambda_k$ for $j \neq k$. Under this assumption, note that the average is given by

$$p_0 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} p(t)\,\mathrm{d}t.$$

Now set

$$\begin{aligned}\|p\|^2 &:= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |p(t)|^2\,\mathrm{d}t \\ &= \sum_{j=-\infty}^{+\infty} |p_j|^2,\end{aligned}$$

and note that term-by-term differentiation yields

$$\|p^{(\ell)}\|^2 = \sum_{j=-\infty}^{+\infty} \lambda_j^{2\ell} |p_j|^2.$$

We now have the following preliminary estimate, which is useful in determining if an almost-periodic function is uniformly bounded.

Lemma 6.8. *If*

$$\sum_{j\neq 0} \frac{1}{\lambda_j^2} < \infty, \tag{6.26}$$

then for some $C \in \mathbb{R}^+$,

$$\sup_{t\in\mathbb{R}} |p(t)| \leq C(\|p\| + \|p'\|).$$

Proof: First note that $|p_0|^2 \leq \|p\|^2$. Upon using the representation of $p(t)$, one then sees that

$$\begin{aligned}|p(t)| &\leq |p_0| + \sum_{j\neq 0} \frac{1}{|\lambda_j|}|\lambda_j p_j| \\ &\leq \|p\| + \left[\sum_{j\neq 0} \frac{1}{\lambda_j^2}\right]^{1/2} \left[\sum_{j=-\infty}^{+\infty} \lambda_j^2 |p_j|^2\right]^{1/2}.\end{aligned}$$

The second line follows from the Cauchy-Schwarz inequality. □

Now consider

$$\begin{aligned}g(t) &:= \int_0^t (p(s) - p_0)\,\mathrm{d}s \\ &= -\mathrm{i}\left(\sum_{j\neq 0} \frac{p_j}{\lambda_j}\mathrm{e}^{\mathrm{i}\lambda_j t} - \sum_{j\neq 0} \frac{p_j}{\lambda_j}\right) \\ &= g_0 + \sum_{j\neq 0} g_j \mathrm{e}^{\mathrm{i}\lambda_j t},\end{aligned}$$

where the coefficients g_ℓ are implicitly defined. If one assumes the bound presented in equation (6.26), then an application of the Cauchy-Schwarz inequality yields that

$$|g_0| \leq \left[\sum_{j\neq 0} \frac{1}{\lambda_j^2}\right]^{1/2} \left[\sum_{j=-\infty}^{+\infty} |p_j|^2\right]^{1/2} \leq C\|p\|;$$

hence, the average of $g(t)$ is well-defined. Similarly, if one assumes that there is an $\epsilon > 0$ such that $|\lambda_j| \geq \epsilon$ for all $j \in \mathbb{Z}\backslash\{0\}$, one sees that

$$\sum_{j\neq 0} |g_j|^2 \leq \frac{1}{\epsilon}\sum_{j\neq 0}|p_j|^2.$$

If one again assumes equation (6.26), then the existence of such a lower bound is implied. Combining these results yields:

Lemma 6.9. *Suppose that the exponents satisfy the bound in equation (6.26). Then*

$$g(t) := \int_0^t (p(s) - p_0)\,\mathrm{d}s$$

is an almost-periodic function with a representation as given in equation (6.25). Furthermore, there is a $C \in \mathbb{R}^+$ such that

$$\|g\| \leq C\|p\|.$$

6.4.2. The averaging theorem

Let us now consider equation (6.4) under the assumption that $f(x, t, \epsilon) = f(x, t)$, and

$$f(x,t) = \bar{f}(x) + \sum_{j=1}^{\infty} a_j(x)\cos(\lambda_j t) + b_j(x)\sin(\lambda_j t), \quad \lambda_j \in \mathbb{R}^+, \tag{6.27}$$

where

$$\bar{f}(x) := \lim_{T\to+\infty} \frac{1}{T}\int_0^T f(x,t)\,\mathrm{d}t. \tag{6.28}$$

Thus, f is assumed to be almost-periodic in $t \in \mathbb{R}^+$, and that it is being written using the notation of real-valued functions. It will further be assumed that $\lambda_j \neq \lambda_k$ for $j \neq k$, and that equation (6.26) holds true.

Define the local average f_T of f by

$$f_T(x,t) := \frac{1}{T}\int_0^T f(x, t+s)\,\mathrm{d}s,$$

and note that if f is T-periodic, then $f_T = \bar{f}$ [**26**, Lemma 3.2.3]. In fact, we have more:

Lemma 6.10. *Under the above assumptions on $f(x, t)$ one has that*

$$f_T(x,t) = \bar{f}(x) + \mathcal{O}(1).$$

Proof: One has that

$$\begin{aligned} f_T(x,t) - \bar{f}(x) &= \frac{1}{T}\int_0^T [f(x,t+s) - \bar{f}(x)]\,\mathrm{d}s \\ &= \frac{1}{T}\int_t^{t+T} [f(x,s) - \bar{f}(x)]\,\mathrm{d}s. \end{aligned}$$

By Lemma 6.9 one has that $f(x,t) - \bar{f}(x)$ is almost-periodic, and hence uniformly bounded, so that

$$|f_T(x,t) - \bar{f}(x)| \leq C.$$

□

If one considers a Lipschitz continuous map $\phi : \mathbb{R} \mapsto \mathbb{R}^n$ with Lipschitz constant L, then one has that

$$\begin{aligned} |\phi(t) - \phi_T(t)| &\leq \frac{1}{T}\int_0^T |\phi(t) - \phi(t+s)|\,\mathrm{d}s \\ &\leq \frac{1}{T}\int_0^T Ls\,\mathrm{d}s \leq \frac{1}{2}LT; \end{aligned}$$

hence,

$$\phi(t) = \phi_T(t) + \mathcal{O}(T). \tag{6.29}$$

We now have the following preliminary estimate:

Lemma 6.11. *Consider equation (6.4). If*

$$\phi(t) := \int_0^t f(x(s), s)\,\mathrm{d}s,$$

then

$$\phi_T(t) = \int_0^t f_T(x(s), s)\,\mathrm{d}s + \mathcal{O}(T)$$

for $t = \mathcal{O}(\epsilon^{-1})$.

Proof: Set

$$R_1 := \frac{1}{T}\int_0^T \int_0^s f(x(r), r)\,\mathrm{d}r\mathrm{d}s, \quad R_2 := \frac{1}{T}\int_0^t \int_0^T [f(x(r+s), r+s) - f(x(s), r+s)]\,\mathrm{d}r\mathrm{d}s.$$

By definition, and after some algebraic manipulation, one has that

$$\phi_T(t) = \int_0^t f_T(x(s), s)\,\mathrm{d}s + R_1 + R_2.$$

Now let L be the Lipschitz constant associated with f, and set

$$M := \sup_{x\in U} \sup_{0\le t\le L/\epsilon} |f(x, t)|.$$

An easy estimate shows that

$$|R_1| \le \frac{1}{T}\int_0^T \int_0^s M\,\mathrm{d}r\mathrm{d}s \le \frac{1}{2}MT,$$

while

$$\begin{aligned} |R_2| &\le \frac{L}{T}\int_0^t \int_0^T |x(r+s) - x(s)|\,\mathrm{d}r\mathrm{d}s \\ &\le \epsilon\frac{L}{T}\int_0^t \int_0^T \int_s^{r+s} |f(x(y), y)|\,\mathrm{d}y\mathrm{d}r\mathrm{d}s \\ &\le \epsilon\frac{L}{T}\int_0^t \int_0^T Mr\,\mathrm{d}r\mathrm{d}s = \frac{1}{2}\epsilon LMTt. \end{aligned}$$

The result now follows. □

We can now prove the quasi-periodic averaging theorem.

Theorem 6.12 (Quasi-periodic Averaging Theorem)**.** *Consider equation (6.4), where $f(x, t)$ is given as in equation (6.27). Let $y(t)$ be the solution to the averaged system*

$$\dot{y} = \epsilon\bar{f}(y), \quad y(0) = x(0),$$

where $\bar{f}$ is defined in equation (6.28). One then has that

$$x(t) = y(t) + \mathcal{O}(\epsilon^{1/2})$$

on the time scale $t = \mathcal{O}(\epsilon^{-1/2})$.

Proof: The solution to equation (6.4) is given by

$$x(t) = x(0) + \epsilon \int_0^t f(x(s), s)\,\mathrm{d}s.$$

Set

$$\phi(t) := \int_0^t f(x(s), s)\,\mathrm{d}s.$$

By equation (6.29) one has that

$$\phi(t) = \phi_T(t) + \mathcal{O}(T),$$

and by Lemma 6.11 one has that

$$\phi_T(t) = \int_0^t f_T(x(s), s)\,\mathrm{d}s + \mathcal{O}(T).$$

Hence,

$$\phi(t) = \int_0^t f_T(x(s), s)\,\mathrm{d}s + \mathcal{O}(T),$$

so that

$$x(t) = x(0) + \epsilon \int_0^t f_T(x(s), s)\,\mathrm{d}s + \mathcal{O}(\epsilon T).$$

Now, the solution to the averaged equation is

$$y(t) = x(0) + \epsilon \int_0^t \bar{f}(y(s))\,\mathrm{d}s,$$

so that

$$\begin{aligned}|x(t) - y(t)| &\le \epsilon \int_0^t |f_T(x(s), s) - \bar{f}(y(s))|\,\mathrm{d}s + \mathcal{O}(\epsilon T)\\ &\le \epsilon \int_0^t |f_T(x(s), s) - f_T(y(s), s)|\,\mathrm{d}s + \epsilon \int_0^t |f_T(y(s), s) - \bar{f}(y(s))|\,\mathrm{d}s + \mathcal{O}(\epsilon T).\end{aligned}$$

As an application of Lemma 6.10 one then has that

$$|x(t) - y(t)| \le \epsilon \int_0^t |f_T(x(s), s) - f_T(y(s), s)|\,\mathrm{d}s + \mathcal{O}(\epsilon t) + \mathcal{O}(\epsilon T).$$

Using the Lipschitz continuity of f_T and Gronwall's inequality then yields that

$$|x(t) - y(t)| = [\mathcal{O}(\epsilon t) + \mathcal{O}(\epsilon T)]\mathrm{e}^{\epsilon L t}.$$

Assuming that $t, T = \mathcal{O}(\epsilon^{-1/2})$ finally gives the desired result. □

It is clear that the result of Theorem 6.12 is not as strong as that of Theorem 6.1. The necessary refinement of Theorem 6.1 is given in [**26**, Lemma 3.7.6], and will be stated below as an unproven corollary.

Corollary 6.13. *Consider equation (6.4), where $f(x, t)$ is given as in equation (6.27). Let $y(t)$ be the solution to the averaged system*

$$\dot{y} = \epsilon \bar{f}(y), \quad y(0) = x(0),$$

where $\bar{f}$ is defined in equation (6.28). One then has that

$$x(t) = y(t) + \mathcal{O}(\epsilon)$$

on the time scale $t = \mathcal{O}(\epsilon^{-1})$.

Remark 6.14. The theory presented above is far from complete. A more recent article which discusses both first- and second-order averaging in the quasi-periodic case is [**2**].

6.4.3. Example: Bose-Einstein condensates

Recall equation (6.14)

$$\begin{aligned}\dot\rho_0 &= -2\delta R^2\rho_0(1-\rho_0)\left[\sin 2\psi\cos\Delta\mu t-\cos 2\psi\sin\Delta\mu t\right]\\ \dot\psi &= -\epsilon(\alpha_{11}^{\mathrm{p}}-\alpha_{00}^{\mathrm{p}})p(t)+\delta R^2\left(\alpha_{111}^1-2\alpha_{011}^0+(4\alpha_{011}^0-\alpha_{000}^0-\alpha_{111}^1)\rho_0\right.\\ &\quad\left.+\,\alpha_{011}^0(2\rho_0-1)\left[\sin 2\psi\sin\Delta\mu t+\cos 2\psi\cos\Delta\mu t\right]\right),\end{aligned}\tag{6.30}$$

and as in the case of the Feshbach resonance assume that $\epsilon=0$, and that $\delta(t+T)=\delta(t)$ for some $T\in\mathbb{R}^+$. Again assuming that $\delta(t)$ is even with zero average, one has that

$$\delta(t)=\sum_{k=1}^{\infty}\delta_k\cos(\frac{k\Delta\mu}{\alpha}t),\quad \alpha\in\mathbb{R}^+.$$

Note that

$$\alpha=\frac{\Delta\mu}{2\pi}T.$$

We wish to apply Corollary 6.13 to equation (6.30). The exponents associated with the quasi-periodic expansion of the vector field are given by

$$\lambda_j=\Delta\mu\left(1+\frac{j}{\alpha}\right),\quad j\in\mathbb{Z}.$$

Hence, equation (6.26) is satisfied if $\alpha\neq j$ for $j\in\mathbb{N}_0$. This condition is automatically satisfied if T is not integrally related to the natural frequency. The case that T is an integral multiple of the natural frequency has already been considered in Section 6.1.2. Under the above nonresonance condition the averaged equations then become

$$\dot{\bar\rho}_0=0,\quad \dot{\bar\psi}=0,$$

i.e., the dynamics are trivial.

6.5. Subharmonic orbits

As in Section 5.4.3, consider

$$\dot x=f(x)+\epsilon g(x,t),\tag{6.31}$$

where $f:\mathbb{R}^2\mapsto\mathbb{R}^2$ is a smooth Hamiltonian vector field, and $g:\mathbb{R}^2\times\mathbb{R}\mapsto\mathbb{R}^2$ is smooth with $g(x,t+T)=g(x,t)$ for some $T>0$. For the case of simplicity, assume that for $x=(u,v)^{\mathrm{T}}$ and $f=(f_1,f_2)^{\mathrm{T}}$ that

$$f_1=\frac{\partial H}{\partial v},\quad f_2=-\frac{\partial H}{\partial u}$$

for a smooth Hamiltonian $H(u,v)$.

Assume that when $\epsilon=0$,

(a) there exists hyperbolic critical point p_0 which is a saddle point

(b) there exists an orbit $q_0(t)$ homoclinic to p_0

Let

$$\Gamma_0:=\{q_0(t)\,:\,t\in\mathbb{R}\}\cup\{p_0\}.$$

Further assume that

(a) the interior of Γ_0 is filled with a continuous family of periodic orbits $q_\alpha(t)$, $\alpha\in(-1,0)$; furthermore, if one sets

$$d(x,\Gamma_0):=\inf_{q\in\Gamma_0}|x-q|,$$

then we have that

$$\lim_{\alpha\to0^-}\sup_{t\in\mathbb{R}}d(q_\alpha(t),\Gamma_0)=0$$

(b) if T_α is the period of q_α and $h_\alpha := H(q_\alpha(t))$, then $T_\alpha(h_\alpha)$ is differentiable with
$$\frac{\mathrm{d}T_\alpha}{\mathrm{d}h_\alpha} > 0.$$

Note that the above two assumptions imply that $T_\alpha \to +\infty$ monotonically as $\alpha \to 0^-$.

Now let $T_\alpha = mT/n$, where $m \in \mathbb{N}$ and $n \in \mathbb{N}$ are relatively prime. If $n = 1$ the orbit is known as a subharmonic orbit, and if $n \geq 2$ the orbit is known as an ultrasubharmonic orbit. Note that q_α is also mT-periodic for any $n \in \mathbb{N}$. If $p_\alpha := q_\alpha(t_0)$ for some $t_0 \in [0, T_\alpha)$, then for the unperturbed Poincaré map Π_0 one has the periodic orbit
$$\{p_\alpha, \Pi_0(p_\alpha), \dots, \Pi_0^{m-1}(p_\alpha)\},$$
i.e., $\Pi_0^m(p_\alpha) = p_\alpha$. Thus, we can consider the persistence and stability of these orbits for the perturbed problem.

We will now use a transformation on equation (6.31) which will make the problem more amenable to analysis. As is seen in [**8**, Chapter 10.3], there exists an invertible symplectic transformation to action-angle variables within the interior of Γ_0 given by
$$I = I(u, v), \quad \theta = \theta(u, v),$$
with inverse
$$u = U(I, \theta), \quad v = V(I, \theta).$$
The transformation becomes singular at Γ_0. The coordinates I and θ are nonlinear polar coordinates, and are chosen in such a way that for the unperturbed problem one has that $\dot{I} = 0$ with $\theta \in [0, 2\pi)$. Under this transformation the unperturbed Hamiltonian becomes
$$H(U, V) \equiv H(I),$$
and equation (6.31) can be rewritten as
$$\begin{aligned} \dot{I} &= \epsilon\left(\frac{\partial I}{\partial u} g_1 + \frac{\partial I}{\partial v} g_2\right) := \epsilon F(I, \theta, t) \\ \dot{\theta} &= \frac{\partial H}{\partial I} + \epsilon\left(\frac{\partial \theta}{\partial u} g_1 + \frac{\partial \theta}{\partial v} g_2\right) := \Omega(I) + \epsilon G(I, \theta, t). \end{aligned} \tag{6.32}$$
The new flow is depicted in Figure 4. The quantity $\Omega(I)$ is the angular frequency of the closed orbit with action I and energy $H(I)$. If q_α has period T_α and action I_α, then one has that $\Omega(I_\alpha) = 2\pi/T_\alpha$. Since
$$\frac{\mathrm{d}T}{\mathrm{d}I} = \frac{\mathrm{d}T}{\mathrm{d}h}\frac{\mathrm{d}h}{\mathrm{d}I} > 0,$$
one has that
$$\frac{\mathrm{d}\Omega}{\mathrm{d}I} = -\frac{2\pi}{T^2}\frac{\mathrm{d}T}{\mathrm{d}I} < 0 \tag{6.33}$$
for all $\alpha \in (-1, 0)$.

Now choose a resonant orbit with period mT/n and action $I_{m,n}$. Upon setting $\Omega_{m,n} := \Omega(I_{m,n})$ and $\delta := \epsilon^{1/2}$, perturb from this orbit using the rotation transformation
$$I = I_{m,n} + \delta h, \quad \theta = \Omega_{m,n} t + \phi. \tag{6.34}$$
Substituting equation (6.34) into equation (6.32) then yields that
$$\begin{aligned} \dot{h} &= \delta F(I_{m,n}, \Omega_{m,n} t + \phi, t) + \mathcal{O}(\delta^2) \\ \dot{\phi} &= \delta \Omega'(I_{m,n}) h + \mathcal{O}(\delta^2). \end{aligned} \tag{6.35}$$

Now, by the chain rule one has that
$$\frac{\partial I}{\partial u} = \frac{\partial I}{\partial H}\frac{\partial H}{\partial u} = -\frac{1}{\Omega} f_2, \quad \frac{\partial I}{\partial v} = \frac{\partial I}{\partial H}\frac{\partial H}{\partial v} = \frac{1}{\Omega} f_1,$$

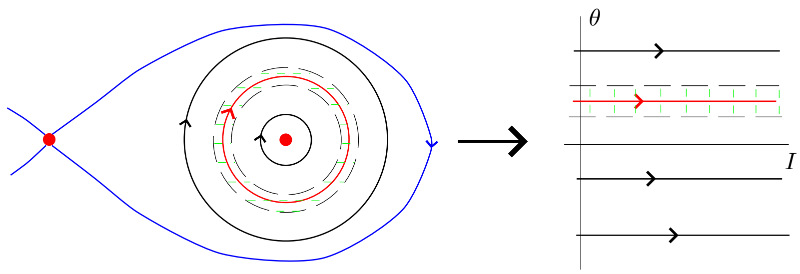

Figure 4: The phase portraits associated with equation (6.31) (left) and equation (6.32) (right). The shaded area represents the region governed by equation (6.35).

so that

$$F(I_{m,n},\Omega_{m,n}t+\phi,t)=\frac{f_1g_2-f_2g_1}{\Omega_{m,n}}.$$

Using the notation of Section 3.1, set

$$M^{m/n}(s):=\int_0^{mT}d(f(q_\alpha(t)),g(q_\alpha(t),t+s))\,\mathrm{d}t. \tag{6.36}$$

The function $M^{m/n}$ is known as the subharmonic Melnikov function (see [**9**, Chapter 4.6] for an alternate derivation). Note that as a consequence of q_α being mT-periodic one has that

$$M^{m/n}(s+mT)=M^{m/n}(s). \tag{6.37}$$

Assuming that F and Ω' are bounded, Theorem 6.1 can be applied to equation (6.35) to get the averaged system

$$\begin{aligned}\dot h&=\frac{\delta}{2\pi n}M^{m/n}\left(\frac{\phi}{\Omega_{m,n}}\right)\\ \dot\phi&=\delta\Omega'(I_{m,n})h.\end{aligned} \tag{6.38}$$

It is important to note that equation (6.38) is a Hamiltonian system with Hamiltonian

$$\bar H(h,\phi):=\delta\left(\frac{1}{2\pi n}\int M^{m/n}\left(\frac{\phi}{\Omega_{m,n}}\right)\,\mathrm{d}\phi-\frac12\Omega'(I_{m,n})h^2\right);$$

hence, the solution structure to equation (6.38) is completely known. Furthermore, as a consequence of equation (6.37) one has that the vector field is maximally $2n\pi$-periodic in ϕ.

Let the zeros of $M^{m/n}$ be denoted by $s_1,\dots,s_k\in(0,mT)$. If the zeros are simple, then as a consequence of the periodicity of $M^{m/n}$ one has that $k=2\ell$ for some $\ell\in\mathbb{N}_0$. The critical points for equation (6.38) are then given by

$$(h,\phi_j)=(0,s_j\Omega_{m,n}),\quad j=1,\dots,k.$$

As a consequence of equation (6.33), one has that the critical point is a saddle if $(M^{m/n})'<0$, and a nonlinear center if $(M^{m/n})'>0$. Note that if the zeros of $M^{m/n}$ are simple, then there is an even number of saddle points and nonlinear centers. The Averaging Theorem 6.1 then implies that there saddle-type orbits to equation (6.31) near the saddle points of equation (6.38), and periodic orbits near the centers, whose stability types are not determined by this $\mathcal{O}(\delta)$ truncation.

Since the unperturbed system is Hamiltonian, one has that the associated Poincaré map Π_0 is area preserving, and hence $\det(\mathrm{D}\Pi_0)=1$. Now suppose that the perturbation is uniformly damping, and as a consequence satisfies the condition that $\nabla\cdot g<0$. Since the perturbed three-dimensional flow contracts volumes like $\mathrm{e}^{(\nabla\cdot g)t}$, the perturbed Poincaré map Π_ϵ must satisfy $\det(\mathrm{D}\Pi_\epsilon)<1$. Since the eigenvalues $\lambda_j\in\sigma(\mathrm{D}\Pi_\epsilon)$ satisfy $\lambda_1\lambda_2=\det(\mathrm{D}\Pi_\epsilon)$, one then necessarily has that:

(a) all periodic points of Π_ϵ are sinks or saddles

(b) there exist no simple invariant closed curves.

The second conclusion follows from the fact that the interior of a simple invariant closed curve is reduced in area under an application of Π_ϵ. It is clear that an analogous result holds if $\nabla \cdot g > 0$, with "sink" being replaced by "saddle".

Remark 6.15. The interesting case that $\nabla \cdot g \equiv 0$, which implies that Π_ϵ is an area preserving map, will be discussed in Section 6.6.

6.5.1. Example

Consider the system

$$\begin{aligned} \dot{u} &= v[1-(u^2+v^2)] + \epsilon[\alpha u - u(u^2+v^2) + \beta u \cos t] \\ \dot{v} &= -u[1-(u^2+v^2)] + \epsilon[\alpha v - v(u^2+v^2)]. \end{aligned} \tag{6.39}$$

Without loss of generality it can be assumed that $\beta \in \mathbb{R}^+$. The system has the feature that when $\epsilon = 0$ all circles are invariant curves, and the circle of radius one is composed solely of critical points. The Hamiltonian associated with the unperturbed problem is given by

$$H(u,v) = \frac{1}{2}(u^2+v^2) - \frac{1}{4}(u^2+v^2)^2.$$

Under the transformation to action-angle variables,

$$u := \sqrt{2I}\,\sin\theta, \quad v := \sqrt{2I}\,\cos\theta,$$

equation (6.39) becomes

$$\begin{aligned} \dot{I} &= \epsilon[2\alpha I - 4I^2 + 2\beta I \sin^2\theta \cos t] \\ \dot{\theta} &= (1-2I) + \epsilon\beta \sin\theta\cos\theta\cos t. \end{aligned} \tag{6.40}$$

The unperturbed Hamiltonian is now given by

$$H(I) = I - I^2,$$

and the period of the unperturbed orbits is

$$T(I) = \frac{2\pi}{1-2I}.$$

Let us study perturbations of the resonant orbit of period 4π with action $I = 1/4$. As in equation (6.34), make the transformation

$$I := \frac{1}{4} + \delta h, \quad \theta := \frac{1}{2}t + \phi \qquad (\delta := \epsilon^{1/2}).$$

After some trigonometrical expansion one sees that the transformed equation is given by

$$\begin{aligned} \dot{h} &= \frac{1}{4}\delta\left[2\alpha - 1 + \beta\left(\cos t + \frac{1}{2}\sin 2\phi\sin t - \frac{1}{2}(1+\cos 2t)\cos 2\phi\right)\right] \\ &\quad + \delta^2\left[2\alpha - 2 + \beta\left(\cos t + \frac{1}{2}\sin 2\phi\sin t - \frac{1}{2}(1+\cos 2t)\cos 2\phi\right)\right] h \\ &\quad + 4\delta^3 h^2 \\ \dot{\phi} &= -2\delta h + \frac{1}{4}\delta^2\left(\cos 2\phi\sin 2t + (1+\cos 2t)\sin 2\phi\right). \end{aligned} \tag{6.41}$$

Note that, unlike equation (6.35), the explicit form of the $\mathcal{O}(\delta^2)$ terms in equation (6.41) are known.

In order to average equation (6.41), use the transformations given in equation (6.5) and equation (6.7) to get

$$h \mapsto h + \frac{1}{4}\delta\beta\left(\sin t - \frac{1}{4}\sin 2\phi\cos 2t - \frac{1}{4}\cos 2\phi\sin 2t\right), \quad \phi \mapsto \phi.$$

After a great deal of calculation it is seen that the averaged equations up to $\mathcal{O}(\delta^3)$ are given by

$$\begin{aligned} \dot{h} &= \frac{1}{4}\delta\left[2\alpha - 1 - \frac{1}{2}\beta\cos 2\phi\right] \\ &\quad + 2\delta^2\left[\alpha - 1 - \frac{1}{4}\beta\cos 2\phi\right] + \mathcal{O}(\delta^3) \\ \dot{\phi} &= -2\delta h + \frac{1}{4}\delta^2\beta\sin 2\phi + \mathcal{O}(\delta^3) \end{aligned} \tag{6.42}$$

(I believe that there is an error in [**9**, equation (4.7.18)]). Note that equation (6.42) can be considered on the torus $(h, \phi) \in \mathbb{R} \times [0, \pi)$.

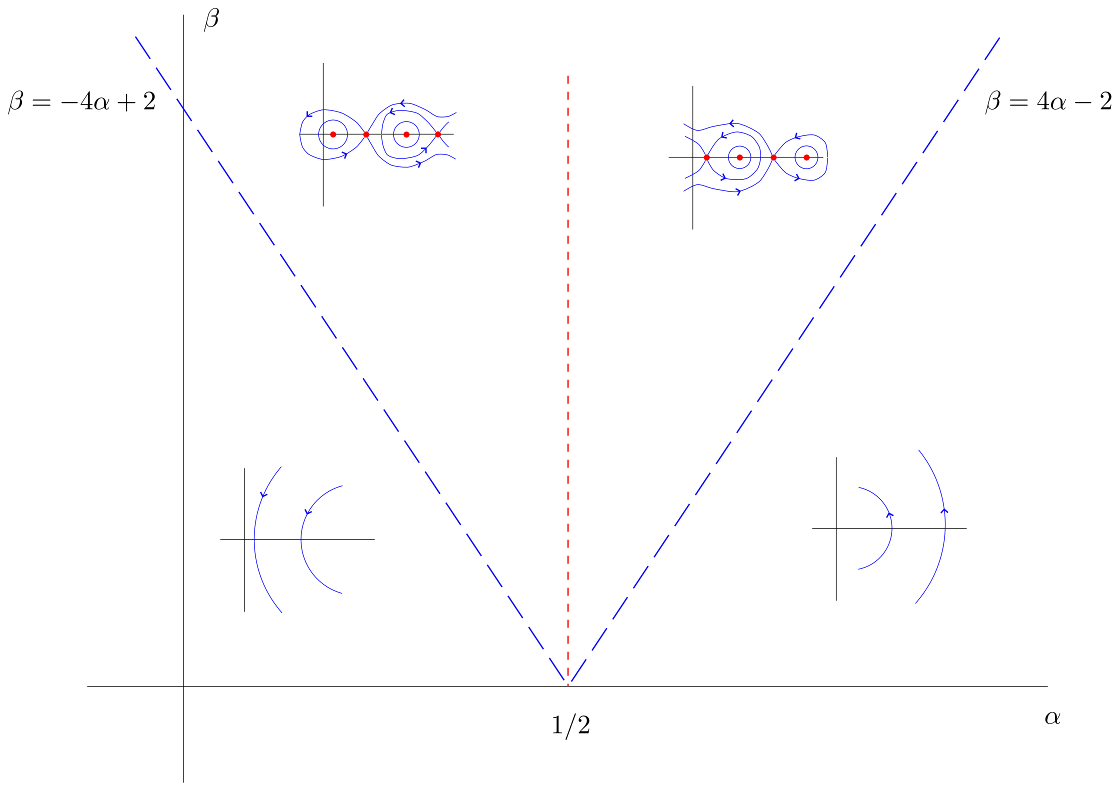

Figure 5: The phase portraits associated with the $\mathcal{O}(\delta)$ averaged equation (6.42).

The Hamiltonian associated with the $\mathcal{O}(\delta)$ terms in equation (6.42) is given by

$$\overline{H}(h, \phi) := \frac{1}{4}\delta\left[(2\alpha - 1)\phi - \frac{1}{4}\beta\sin 2\phi + 4h^2\right].$$

Note that if $\alpha = 1/2$, then $\overline{H}$ is π-periodic in ϕ; hence, one may expect this line to be a bifurcation line in parameter space. If

$$\beta > 2|2\alpha - 1|,$$

then there will exist two critical points for $\phi \in [0, \pi)$; otherwise, there will exist none. The critical point satisfying $\sin 2\phi < 0$ will be a saddle point, and the other will be a nonlinear center. Upon reviewing equation (6.39), it is seen that

$$\nabla \cdot g = 2\alpha + \beta \cos t - 4(u^2 + v^2);$$

hence, if $\beta < 2|\alpha|$ and $\alpha \in \mathbb{R}^-$ the nonlinear center will become a sink at $\mathcal{O}(\delta^2)$. Note, however, that no critical points exist which satisfy this condition. Sample phase portraits at $\mathcal{O}(\delta)$ are given in Figure 5. The interested student should consult [**9**, Figure 4.7.2] to see the phase portraits at $\mathcal{O}(\delta^2)$. It is an interesting exercise to fully develop the bifurcation diagram.

6.6. KAM and Twist theorems

As a consequence of the discussion in Section 6.5, the leading order effect of resonant perturbations is now understood. In this section we will undertake the study of those orbits which are not forced at resonance.

6.6.1. Algebraic preliminaries

The following material can be found [**10**]. Let $\theta \in \mathbb{R}$ be such that it is a root of a polynomial equation with integral coefficients. Such a number θ is said to be an algebraic number. If θ satisfies an equation of degree n, but none of lower degree, then it is said to be an algebraic number of degree n. Set

$$\mathcal{A}_n := \{\theta : \theta \text{ is an algebraic number of degree } n\}.$$

It is easy to see that $\theta \in \mathcal{A}_1$ implies that $\theta \in \mathbb{Q}$. In fact, $\mathcal{A}_1 = \mathbb{Q}$, so that $\mathcal{A}_1$ is a countable set. It is also easy to see that if $q \in \mathcal{A}_1$, then $\pm|q|^{1/n} \in \mathcal{A}_n$ for each $n \geq 2$. It can be shown that $\mathcal{A}_n$ is countable for each $n \geq 1$; hence, the set

$$\mathcal{A} := \bigcup_{n=1}^{\infty} \mathcal{A}_n,$$

i.e., the set of algebraic numbers, is itself countable and has Lebesgue measure zero. One has that θ is transcendental if $\theta \notin \mathcal{A}$. As a consequence, almost all real numbers are transcendental.

Algebraic numbers have the property that they cannot be approximated "too rapidly" by rational numbers. A number θ is said to be approximated to degree k if the inequality

$$\left|\frac{p}{q} - \theta\right| < |q|^{-k}$$

has an infinite number of solutions $p, q \in \mathbb{Z}$. It can be shown that if $\theta \in \mathbb{Q}$, then θ can be approximated to order one, and to no higher order. Furthermore, if $\theta \notin \mathbb{Q}$, then θ can be approximated to order two. The following theorem, which is due to Liouville, gives a characterization as to how well the algebraic numbers can be approximated.

Theorem 6.16. *If $\theta \in \mathcal{A}_n$ for some $n \in \mathbb{N}$, then θ cannot be approximated to any order greater than n.*

Remark 6.17. Two direct consequences of Liouville's theorem are:

(a) if $\theta \in \mathcal{A}_n$, then there exists an $\epsilon > 0$ such that

$$\left|\frac{p}{q} - \theta\right| \geq \epsilon |q|^{-n}, \quad p, q \in \mathbb{Z} \tag{6.43}$$

(b) if $\theta \notin \mathcal{A}$, then it can be approximated by rational numbers arbitrarily quickly.

6.6.2. KAM and Twist theorems

Regarding equation (6.31), suppose now that the flow is volume-preserving, i.e., $\nabla\cdot g\equiv 0$. Once again consider the transformed equation (6.32), and note that the unperturbed problem is

$$\dot{I}=0,\quad \dot{\theta}=\Omega(I), \tag{6.44}$$

where $\Omega(I_\alpha)=2\pi/T_\alpha$. The Poincaré map associated with equation (6.44) is given by

$$\begin{aligned}\Pi_0(I,\theta)&=(I,\theta+\Omega(I)T)\\&=\left(I,\theta+2\pi\frac{T}{T_\alpha}\right).\end{aligned} \tag{6.45}$$

Since $\Omega'(I)\neq 0$ (recall that $\Omega'(I)<0$), the map Π_0 is known as a twist map. Note that it preserves area. Finally note that if $T/T_\alpha\in\mathbb{Q}$, then the associated orbit is periodic; otherwise, it densely fills the circle $I=I_\alpha$. The perturbed mapping is given by

$$\Pi_\epsilon(I,\theta)=(I+\epsilon f(I,\theta),\Omega(I)T+\epsilon g(I,\theta)), \tag{6.46}$$

where f and g are bounded and 2π-periodic in θ. Since the flow is volume-preserving, the map Π_ϵ is area-preserving. As a side remark, since the perturbed map is area-preserving one has that any elliptic points are actually centers. One has the following important result concerning the existence of tori.

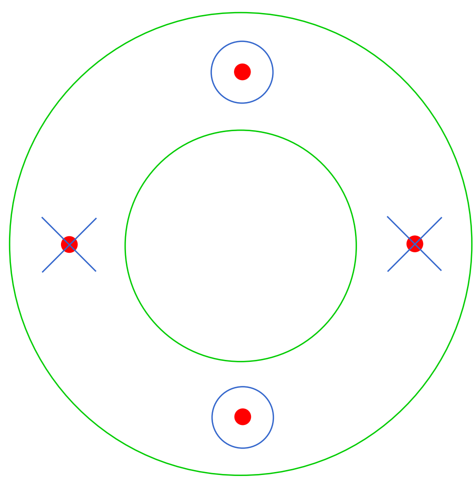

Figure 6: The Poincaré map associated with equation (6.46). The green curves are invariant curves whose existence is guaranteed by Theorem 6.19. The fixed points, and their stability type, are guaranteed by the discussion in Section 6.5.

Theorem 6.18 (KAM)**.** *If $\Omega'(I)\neq 0$ and $\epsilon>0$ is sufficiently small, then Π_ϵ has a set of invariant closed curves of positive Lebesgue measure close to the set $I=I_\alpha$. The surviving closed curves are filled with dense irrational orbits.*

Unfortunately, Theorem 6.18 does not yield any information as to which of the unperturbed closed curves persist. As a consequence of the discussion in Section 6.5, we know that this set does not include any unperturbed curve which satisfies $T/T_\alpha\in\mathbb{Q}$. The following result fills in this gap.

Theorem 6.19 (Twist Theorem)**.** *Under the assumptions of Theorem 6.18, if $T/T_\alpha\in\mathcal{A}_n$ for some $n\geq 2$, then the invariant curve I_α persists as an invariant closed curve for equation (6.46). Furthermore, the curve is filled with dense irrational orbits.*

The persistence of the invariant closed curves has an important implication regarding the behavior of the mapping Π_ϵ. Let γ_1 and γ_2 be two such invariant curves, and assume that $\gamma_1 \subset \mathrm{int}(\gamma_2)$. Set the annulus

$$\mathcal{A}_\epsilon := \mathrm{ext}(\gamma_1) \cap \mathrm{int}(\gamma_2),$$

and suppose that $p \in \mathcal{A}_\epsilon$ (see Figure 6). Since the map Π_ϵ is induced by the flow of equation (6.31), as a consequence of uniqueness of solutions to equation (6.31) one must that $\Pi_\epsilon(p) \in \mathcal{A}_\epsilon$. In other words, $\Pi_\epsilon(\mathcal{A}_\epsilon) \subset \mathcal{A}_\epsilon$. Since $\Omega'(I) \neq 0$, there will exist an initially resonant orbit within $\mathcal{A}_\epsilon$. As a consequence of the discussion in Section 6.5 one knows that this resonant orbit breaks into an even number of fixed points, half of which are centers and half of which are saddle points. The invariant manifolds of the saddle points are contained within $\mathcal{A}_\epsilon$, and they must intersect (otherwise, Π_ϵ would not preserve area). In general, one can expect that some of these intersections will be transverse. In this case, the result of Theorem 5.12 is relevant, and consequently one has the existence of homoclinic tangles, and all of the attendant complicated dynamics.

6.7. Example: Bose-Einstein condensates

Assuming that there is no magnetic trap, and after several rescalings, the steady-state problem associated with equation (1.1) in the repulsive case can be written as

$$\ddot{q} + q - q^3 = \epsilon\cos(\kappa x)q \tag{6.47}$$

(see [25] for the details). The periodic solutions to equation (6.47) are given by

$$q(x,k) := \sqrt{\frac{2k^2}{1+k^2}}\,\mathrm{sn}(z,k), \quad z := \frac{x}{\sqrt{1+k^2}},$$

where $k \in (0,1)$, and satisfy the initial condition

$$q(0,k) = 0, \quad \dot{q}(0,k) = \sqrt{2}\frac{k}{1+k^2}.$$

The period of $q(x,k)$ is

$$T(k) := 4\sqrt{1+k^2}K(k),$$

where $K(k)$ is the elliptic integral of the first kind, i.e.,

$$K(k) := \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2\phi}}\,\mathrm{d}\phi.$$

The period has the property that $T'(k) > 0$, with

$$\lim_{k\to 0^+} T(k) = 2\pi, \quad \lim_{k\to 1^-} T(k) = +\infty.$$

Finally, $q(x,k)$ is odd in x.

Upon following the discussion in Section 6.5, one has that for the unperturbed problem the action is given by

$$\begin{aligned} I(k) &:= \frac{1}{2\pi}\int_0^{T(k)} \dot{q}(x,k)^2\,\mathrm{d}x \\ &= \frac{4}{3\pi(1+k^2)^{3/2}}\left\{(1+k^2)E(k) - (1-k^2)K(k)\right\}, \end{aligned}$$

where $E(k)$ is the elliptic integral of the second kind, i.e.,

$$E(k) := \int_0^{\pi/2} \sqrt{1-k^2\sin^2\phi}\,\mathrm{d}\phi.$$

The evaluation of the integral follows from the identities given in [24, Chapter 4.10]. The associated angle for the unperturbed problem is given by

$$\theta(x,k) := \theta(0) + \frac{2\pi}{T(k)}x.$$

The space-dependent Hamiltonian associated with equation (6.47) is given by

$$\mathcal{H}(q,\dot{q},x) := \frac{1}{2}\dot{q}^2 + \frac{1}{2}q^2 - \frac{1}{4}q^4 + \frac{1}{2}\epsilon q^2\cos(\kappa x).$$

Although it will not be done herein, as it is not necessary for the subsequent analysis, one can now compute the Hamiltonian in action-angle variables (see [25, equation (43)]).

Remark 6.20. Although it will not be pursued herein, this exact evaluation of the Hamiltonian in action-angle variables allows one to "easily" apply a higher order perturbation theory. This avenue of attack is pursued in [25].

As seen in Section 6.5, the important quantity to calculate is the Melnikov function given in equation (6.36). One has a resonance if for $m, n \in \mathbb{N}$,

$$\kappa = \frac{m}{n}\frac{\pi}{2K(k)}\frac{1}{\sqrt{1+k^2}}. \tag{6.48}$$

For ease, now assume that $n = 1$. Upon using the definition in equation (6.36), the restriction of equation (6.48), and the identity

$$\mathrm{sn}(u,k)^2 + \mathrm{cn}(u,k)^2 = 1,$$

one sees that the Melnikov function associated with equation (6.47) is given by

$$M^m(\phi) := -\frac{k^2}{1+k^2}\frac{m\pi}{2K(k)}M_c(k,m)\sin(m\phi), \tag{6.49}$$

where

$$M_c(k,m) := \int_{-2mK(k)}^{2mK(k)} \mathrm{cn}(u,k)^2\cos\left(\frac{m\pi}{2K(k)}u\right)\,\mathrm{d}u. \tag{6.50}$$

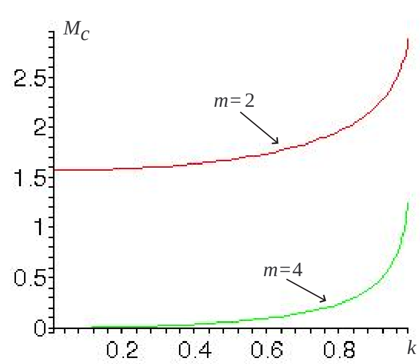

Figure 7: A plot of the constant $M_c(k,m)$ defined in equation (6.50) for $m = 2, 4$.

A plot of the constant $M_c(k,m)$ for $m = 2, 4$ is given in Figure 7. First, as a consequence of the fact that the Fourier cosine series for $\mathrm{cn}(u,k)^2$ is given by

$$\mathrm{cn}(u,k)^2 = \mathcal{C}_0(k) + \sum_{n=1}^{\infty}\mathcal{C}_n(k)\cos\left(\frac{2n\pi}{2K(k)}u\right)$$

[**25**, equation (52)], one has that $M_c(k, 2\ell+1) \equiv 0$ for $\ell \in \mathbb{N}_0$. Thus, if $m = 2\ell+1$, the theory presented in Section 6.5 does not give a definitive answer, and a higher-order Melnikov theory is necessary (see [**24**, Chapter 4.11] for some details). As a consequence, it will henceforth be assumed that $m = 2\ell$ for some $\ell \in \mathbb{N}$ (this assumption is also taken in [**25**]). As a side remark, since $\mathrm{cn}(u, k)^2$ is holomorphic on a strip containing the real axis, one has that for fixed $k \in (0, 1)$ and any $N \in \mathbb{N}$,

$$\lim_{\ell\to+\infty} M_c(k, 2\ell)(2\ell)^N = 0.$$

Hence, one must also have that the upper bound on ϵ for which the theory presented in Section 6.5 is applicable approaches zero as $\ell \to +\infty$. The primary point of [**25**] is to resolve this limit. In any event, one finally gets that

$$M^{2\ell}(\phi) = -\frac{k^2}{1+k^2}\frac{\ell\pi}{K(k)} M_c(k, 2\ell)\sin(2\ell\phi).$$

Appealing to equation (6.38) and the subsequent discussion, one sees that the critical points for the averaged equation in the resonance band are given by

$$\phi_n := \frac{n}{2\ell}\pi, \quad n = 0, \ldots, 4\ell-1.$$

Thus, the resonance band has 2ℓ centers and 2ℓ saddles, and the centers correspond to those points in which n is odd, and the centers correspond to those points for which n is even.

Remark 6.21. The physical interpretation of this result is that as the frequency of the optical lattice increases, the number of critical points within a particular resonance band increases. Thus, increasing the frequency of the optical lattice increases the complexity of the dynamics for the steady-state problem.

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