MATH W81: HOMEWORK #6

1. Let

$$\boldsymbol{u}_1 = \frac{1}{\sqrt{14}}\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \quad \boldsymbol{u}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \boldsymbol{u}_3 = \frac{1}{\sqrt{21}}\begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix}.$$

Let the Hermitian matrix $\boldsymbol{A}$ be given by

$$\boldsymbol{A} = -2\boldsymbol{u}_1\boldsymbol{u}_1^{\mathrm{T}} + \boldsymbol{u}_2\boldsymbol{u}_2^{\mathrm{T}} + 3\boldsymbol{u}_3\boldsymbol{u}_3^{\mathrm{T}}.$$

Set

$$S^{\perp} = \mathrm{Span}\{\boldsymbol{s}^{\perp}\}, \quad \boldsymbol{s}^{\perp} = \frac{1}{\sqrt{14}}\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}.$$

(a) Let $S = (S^{\perp})^{\perp}$. Find a projection matrix $\boldsymbol{P}$ with the properties:

- $\boldsymbol{P} : \mathbb{C}^3 \mapsto S$
- $\boldsymbol{P}^2\boldsymbol{x} = \boldsymbol{P}\boldsymbol{x}$ for any $\boldsymbol{x} \in \mathbb{C}^3$
- $\boldsymbol{P}\boldsymbol{x} = \boldsymbol{0}$ for all $\boldsymbol{x} \in S^{\perp}$.

(b) Find the 2×2 Hermitian matrix representation $\boldsymbol{A}_{\mathrm{rep}}$ for the linear operator $\boldsymbol{PAP} : S \mapsto S$.

(c) Explicitly construct a function, say $r(\lambda)$, which has the properties:

- $r(\lambda) = 0$ if and only if $\lambda \in \sigma\left(\boldsymbol{A}_{\mathrm{rep}}\right)$
- the graph of $r(\lambda)$ has vertical asymptotes for $\lambda \in \sigma(\boldsymbol{A})$.

(d) Analyze the graph of $r(\lambda)$, and from this analysis explicitly state how the eigenvalues of $\boldsymbol{A}_{\mathrm{rep}}$ relate to those of $\boldsymbol{A}$.

2. Consider the generalized eigenvalue problem

$$\begin{pmatrix} a & 1 & 2 \\ 1 & 3 & 5 \\ 2 & 5 & 3 \end{pmatrix}\boldsymbol{v} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\boldsymbol{v}.$$

Construct a function, say $r(\lambda)$, with the properties:

- $r(\lambda) = 0$ implies that λ is an eigenvalue
- the graph of $r(\lambda)$ has two vertical asymptotes.

Analyze the graph of $r(\lambda)$ in order to answer the following questions:

(a) For which values of λ does $r(\lambda)$ have vertical asymptotes?

(b) For which value(s) of a does $r(\lambda) = 0$ have three distinct real-valued solutions?

(c) When $r(\lambda) = 0$ has three real-valued solutions, how does the location of these zeros relate to the location of the vertical asymptotes?

(d) For which value(s) of a does $r(\lambda) = 0$ have only one real-valued solution? What can be said about the eigenvalues in this case? What can be said about the location of the one real-valued zero relative to the location of the vertical asymptotes?