MATH W81: HOMEWORK #1

1. Consider the homogeneous system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$, where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ with $m < n$. In other words, $\boldsymbol{A}$ is a $m \times n$ matrix with real-valued coefficients. Explain why this system must always have an infinite number of solutions.

2. Consider the nonhomogeneous system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, i.e., $\boldsymbol{A}$ is a $m\times n$ matrix with real-valued coefficients, and $\boldsymbol{b} \in \mathbb{R}^{m}$ is nonzero.

(a) If $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are two solutions, must it be the case that $5\boldsymbol{x}_1 + 7\boldsymbol{x}_2$ is also a solution? Why, or why not?

(b) Suppose that $m < n$, and further suppose that the system is consistent. Is it possible for the solution to be unique? Why, or why not?

3. Find all of the solutions to the system

$$\begin{aligned} x - 3y + z &= -5 \\ x + 2y + 4z &= 5 \\ -3x + 2y - 4z &= 1. \end{aligned}$$

If the system is not consistent, state why.

4. Approximate the Sturm-Liouville problem

$$y'' + r(x)y = \lambda w(x)y, \quad y'(0) = y'(1) = 0,$$

as the generalized eigenvalue problem

$$(\boldsymbol{D}_2 + R)\boldsymbol{y} = \lambda \boldsymbol{W}\boldsymbol{y}.$$

Clearly define what are the matrices $\boldsymbol{D}_2, \boldsymbol{R}, \boldsymbol{W}$ and the vector $\boldsymbol{y}$.

5. For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n$ we have the inner-product

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^{\mathrm{T}}\overline{\boldsymbol{y}}.$$

Show that the inner-product has the properties:

(a) $\langle \boldsymbol{x} + \boldsymbol{y}, \boldsymbol{z} \rangle = \langle \boldsymbol{x}, \boldsymbol{z} \rangle + \langle \boldsymbol{y}, \boldsymbol{z} \rangle$

(b) $\langle \boldsymbol{y}, \boldsymbol{x} \rangle = \overline{\langle \boldsymbol{x}, \boldsymbol{y} \rangle}$

(c) for $a \in \mathbb{C}$, $\langle a\boldsymbol{x}, \boldsymbol{y} \rangle = a\langle \boldsymbol{x}, \boldsymbol{y} \rangle$, and $\langle \boldsymbol{x}, a\boldsymbol{y} \rangle = \overline{a}\langle \boldsymbol{x}, \boldsymbol{y} \rangle$

(d) $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \in \mathbb{R}$

(e) $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \geq 0$, and $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ if and only if $\boldsymbol{x} = \boldsymbol{0}$.