Math 355 Homework Problems #4

MATRIX ANALYSIS AND APPLIED LINEAR ALGEBRA, by C. Meyer

1. Find the matrix representation for the linear transformation $\mathcal{L} : \mathbb{F}_2[x] \mapsto \mathcal{M}_2(\mathbb{F})$ given by,

$$a_0 + a_1 x + a_2 x^2 \mapsto \begin{pmatrix} a_0 - a_1 + a_2 & 2a_0 + 4a_2 \\ -a_0 + 2a_1 & 3a_0 + 5a_1 + 11a_2 \end{pmatrix}.$$

2. Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{F}^n$, $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^{\mathrm{H}} \boldsymbol{y}$. Let a norm on $\mathbb{F}^n$ induced by this inner product be denoted $\| \cdot \|_2$,

$$\|\boldsymbol{x}\|_2^2 = \langle \boldsymbol{x}, \boldsymbol{x} \rangle.$$

A matrix $\boldsymbol{Q} \in \mathcal{M}_n(\mathbb{F})$ is an orthonormal matrix if the columns of $\boldsymbol{Q}$ form an orthonormal basis for $\mathbb{F}^n$. Show that:

(a) $\boldsymbol{Q}^{\mathrm{H}} \boldsymbol{Q} = \boldsymbol{Q} \boldsymbol{Q}^{\mathrm{H}} = \boldsymbol{I}_n$

(b) $\|\boldsymbol{Q}\boldsymbol{x}\|_2 = \|\boldsymbol{x}\|_2$ (the linear transformation preserves norm)

(c) $\langle \boldsymbol{Q}\boldsymbol{x}, \boldsymbol{Q}\boldsymbol{y} \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle$ (the linear transformation preserves angle)

3. Let the inner product on $\mathbb{R}_3[x]$ be given by

$$\langle f, g \rangle = \int_0^1 f(x) g(x) \, \mathrm{d}x.$$

Find the angle between $1 - x$ and x^3.