Math 355 Homework Problems #2

MATRIX ANALYSIS AND APPLIED LINEAR ALGEBRA, by C. Meyer

1. Let

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 4 & -4 \\ 2 & 1 & 5 \end{pmatrix}.$$

Find a nonsingular matrix $\boldsymbol{P}$ such that $\boldsymbol{PA} = \boldsymbol{E_A}$. If you wish, you may write $\boldsymbol{P} = \boldsymbol{E}_k\boldsymbol{E}_{k-1}\cdots\boldsymbol{E}_2\boldsymbol{E}_1$ for some nonsingular matrices $\boldsymbol{E}_j$.

2. The trace of a square matrix $\boldsymbol{A} \in \mathcal{M}_n(\mathbb{R})$, trace($\boldsymbol{A}$), is the sum of the diagonal elements. Let $\boldsymbol{A},\boldsymbol{B} \in \mathcal{M}_n(\mathbb{R})$ and $a \in \mathbb{R}$. Show that with respect to matrix addition and scalar multiplication,

(a) $\operatorname{trace}(a\boldsymbol{A}) = a\operatorname{trace}(\boldsymbol{A})$

(b) $\operatorname{trace}(\boldsymbol{A}+\boldsymbol{B}) = \operatorname{trace}(\boldsymbol{A}) + \operatorname{trace}(\boldsymbol{B})$.

3. Let $\boldsymbol{A},\boldsymbol{B} \in \mathcal{M}_{m\times n}(\mathbb{R})$. Show that with respect to matrix/matrix multiplication,

(c) $\operatorname{trace}\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B}\right) = \sum_{j=1}^{n} \boldsymbol{a}_j^{\mathrm{T}}\boldsymbol{b}_j$, where $\boldsymbol{a}_j$ is the j^{th} column of $\boldsymbol{A}$, and $\boldsymbol{b}_j$ is the j^{th} column of $\boldsymbol{B}$

(d) $\operatorname{trace}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}) = 0$ if and only if $\boldsymbol{A}$ is the zero matrix.

4. Let $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^n$ be given, and set $\boldsymbol{A} = \boldsymbol{x}\boldsymbol{y}^{\mathrm{T}}$ to be the rank-one matrix formed from these vectors. Show that,

$$\boldsymbol{A}^2 = \operatorname{trace}(\boldsymbol{A})\boldsymbol{A}.$$

5. Determine which of the following subsets of $\mathcal{M}_n(\mathbb{R})$ are subspaces of $\mathcal{M}_n(\mathbb{R})$. Provide an explanation for your answer.

(a) all matrices, $\boldsymbol{A}$, such that $\operatorname{trace}(\boldsymbol{A}) = 0$

(b) all Hermitian matrices

(c) all skew-Hermitian matrices

(d) all matrices, $\boldsymbol{A}$, such that $\boldsymbol{A}^2 = \boldsymbol{A}$.

6. Suppose that $n = 2$ in Problem **5**. Find a basis for each set which is a subspace. Determine the dimension of each of these subspaces.